Operator theory on tent spaces

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Résumé

Nous donnons un mécanisme de type Calderón-Zygmund concernant la théorie de l'extrapolation pour des opérateurs d'intégrale singulière sur les espaces de tentes. Pour des opérateurs de régularité maximale sur les espaces de tentes, nous donnons des résultats optimaux en exploitant la structure des opérateurs intégraux de convolution et en utilisant des estimations de la décroissance hors-diagonale du semi-groupe ou de la famille résolvante sous-jacente.

Nous appliquons des techniques précédentes d'analyse harmonique et fonctionnelle pour estimer sur les espaces de tentes certains opérateurs d'intégrale évolutionnelle, nées de l'étude des problèmes aux limites elliptiques et des systèmes non-autonomes du premier ordre.

Mots-clefs : Espaces de tentes, opérateurs d'intégrale singulière, extrapolation, régularité maximale conique, problèmes aux limites elliptiques, systèmes du premier ordre.

OPERATOR THEORY ON TENT SPACES

Abstract

We give a Calderón-Zygmund type machinery concerning the extrapolation theory for the singular integral operators on tent spaces. For maximal regularity operators on tent space, we give some optimal results by exploiting the structure of convolution integral operators and by using the off-diagonal decay estimates of the underlying semigroup or resolvent family.

We apply the previous harmonic and functional analysis techniques to estimate on tent spaces certain evolutionary integral operators arisen from the study of boundary value elliptic problems and first order non-autonomous systems.

Keywords : Tent spaces, singular integral operators, extrapolation, conical maximal regularity, boundary value elliptic problems, first order systems.

獻給青青和我親愛的父母
à Qing-Qing et mes chers parents

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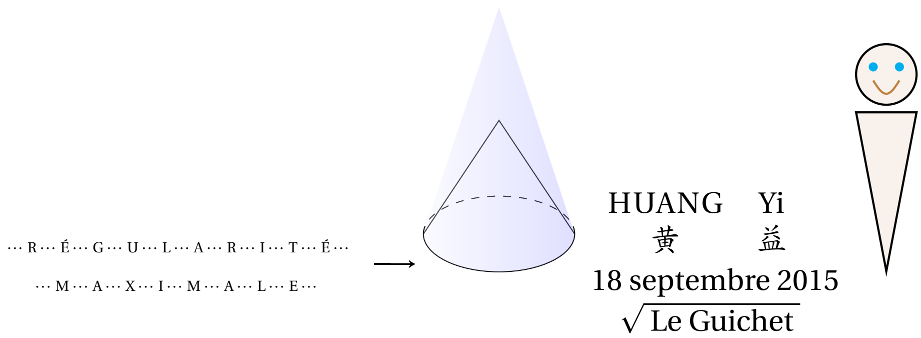

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Guide de Lecture

N.B. 1) La dépendance entre des chapitres

(i) Chapitre 1 est nécessaire pour Chapitre 2.
(ii) Chapitre 4 est nécessaire pour Chapitre 5.
(iii) Chapitre 5 s'inspires de Chapitre 3.

Voir l'Introduction Générale pour la relation précisée.

N.B. 2) Dans chaque chapitre les notations (en particulier lesquels sur des espaces de tentes et la décroissance hors-diagonale) sont rappelés et les bibliographies sont préparés. Par conséquent, ils peuvent être lus séparément.

N.B. 3) Pour la raison de 2), notre index s'affiche que par sa première apparition.

吾生也有涯；而知也無涯. *(La vie humaine est limitée ; le savoir est illimité.)*

莊子 (Tchouang-Tseu) ∼ 369–286 av. J.-C.

Ce que nous connaissons est peu de chose ; ce que nous ignorons est immense.

Pierre-Simon Laplace (∆) 1749–1827

Introduction Générale

Abstract

Nous sommes intéressés dans cette thèse par le problème d'extrapolation des trois types d'opérateurs sur les espaces de tentes. Par ordre, ils sont

Partie I – Opérateurs d'Intégrale Singulière.

Partie II – Opérateurs de Régularité Maximale.

Partie III – Calcul Opérationnel Holomorphe.

Nous développons quelques outils pour les étudier.

------------ Un Petit Plan (Q & R) ------------

Avant de passer à l'aspect technique, je vous invite à quelques questions générales que je me demandais durant la préparation de cette thèse. Je pense que le lecteur voudrait également les connaître. Les réponses ci-dessous sont selon mes compréhensions personnelles. Il y a certainement beaucoup d'autres réponses.

(Q1). Pourquoi avons-nous besoin des espaces de tentes ?

(R). Les espaces de tentes ont été inventés par Coifman, Meyer et Stein dans [CMS83] pour donner une nouvelle preuve du théorème de Coifman, McIntosh et Meyer sur les intégrales de Cauchy. En outre, comme expliqué dans [CMS85], ils fournissent un cadre pour étudier de nombreux objets d'analyse harmonique, par exemple, les fonctions carrés, les espaces de Hardy et la dualité de Carleson.

(Q2). Pourquoi des espaces de tentes avec des moyennes de Whitney ?

(R). Certains espaces fonctionnels avec des moyennes de Whitney ont été introduits par Dahlberg dans [Dah86] et par Kenig et Pipher dans [KP93] dans l'étude des problèmes elliptiques aux limites avec des coefficients peu réguliers. Avec [Dah86, KP93] inclus, nous élargissons le cadre de [CMS85] dans Chapitre 4.

(Q3). Théorie des Opérateurs sur les Espaces de Tentes – motivation et objectif ?

(R). Cela est nécessaire en extrapolant les résultats de la solvabilité obtenus par Auscher et Axelsson dans [AA11] pour des problèmes elliptiques aux limites. Cette extrapolation sera abordée dans Chapitre 5. L'objet central dans [AA11] est un opérateur de régularité maximale associé aux opérateurs de Dirac du premier ordre.

(Q4). Qu'est-ce que nous avons pour les autres opérateurs de régularité maximale sur les espaces de tentes ?

(R). Pour les opérateurs de régularité maximale associés aux opérateurs elliptiques d'ordre deux, nous développons dans Chapitre 3 une théorie de régularité maximale (conique) sur les espaces de tentes. Ceci est plus impliqué que des résultats de régularité maximale (classique) obtenus par Blunck et Kunstmann dans [BK02].

(Q5). Que pouvons-nous attendre pour les opérateurs d'intégrale singulière (modelés sur les opérateurs de régularité maximale) sur les espaces de tentes ?

(R). Il est bien connu que les espaces de tentes sont la réalisation d'extensions harmoniques des espaces de Hardy. On peut s'attendre à une théorie d'opérateurs d'intégrale singulière sur les espaces de tentes. Cette théorie a été partiellement réalisée en [AKMP12]. Nous donnons une théorie de type Calderón-Zygmund dans Chapitre 1 et Chapitre 2, donc étendons [AKMP12]. Cela nous aussi amène à étendre certains résultats classiques de C. Fefferman dans [Fef70] et de C. Fefferman et Stein dans [FS72].

Maintenant, on va commenter en détail par l'ordre de l'apparition des chapitres.

0.1 Opérateurs d'Intégrale Singulière —— Théorie de Calderón-Zygmund

La première partie de cette thèse est consacrée à améliorer les articles : [Aus11], P. Auscher, « Change of angle in tent spaces », C. R. Math. Acad. Sci. Paris, vol. **349**, (2011), et [AKMP12], P. Auscher, C. Kriegler, S. Monniaux et P. Portal, « Singular integral operators on tent spaces », J. Evol. Equ., vol. **12**, (2012).

L'objet que nous intéresse est **l'opérateur d'intégrale singulière** que l'on peut décrire comme suit : T est un opérateur borné sur $L^2(\mathbb{R}_+ \times \mathbb{R}^n) = L^2(\mathbb{R}_+; \mathbb{R}^n)$, où $\mathbb{R}_+ = (0, \infty)$, tel que pour f une fonction à valeurs dans $L^2(\mathbb{R}^n)$ et à support borné dans $\mathbb{R}_+$, et, pour

$t \notin \operatorname{supp} f$, alors l'opérateur T a la représentation

$$T(f)_t = \int_0^t K(t,s) f_s \, ds,$$

où $f_s = f(s,\cdot)$, et $K(t,s)$, $0 < s < t < \infty$, est un noyau satisfaisant pour une constante C

$$\|K(t,s)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leq C \frac{1}{t-s}.$$

On dit que $T \in SIO^+$. Donc la "singularité" pour le noyau de T arrive quand $s = t$.

Le but principal de cette partie, c'est de comprendre les remarques à la fin de l'introduction de l'article [AKMP12] concernant SIO^+ agissant sur les espaces de tentes,

"Calderón-Zygmund theory does not seem to be an appropriate

machinery to study singular integral operators (on tent spaces) ..."

Nous remarquons que l'utilisation des décompositions atomiques des espaces de tentes dans [AKMP12] s'inspire de [Aus11]. Cela nous amène aussi à considérer un problème posé dans [Aus11].

À la recherche de cette théorie de Calderón-Zygmund sur les espaces de tentes, nous bénéficions beaucoup des articles de Blunck et Kunstman concernant la théorie des opérateurs sur les espaces de Lebesgue : [BK02] sur la régularité maximale, [BK03] sur les opérateurs de Calderón-Zygmund, et [BK04] sur la transformée de Riesz.

Chapitre 1. Dans ce chapitre, nous prouvons une certaine décomposition de type Calderón-Zygmund pour les fonctions des espaces de tentes de Coifman-Meyer-Stein. Comme application, nous donnons une preuve unifiée des généralisations dans les espaces de tentes des estimations faibles de type "endpoint" de C. Fefferman pour les "grandes fonctions carrées" et des estimations faibles de type "endpoint" de C. Fefferman et E. M. Stein pour les fonctions maximales de boîtes.

------------ Plus Précisément ------------

Posons $\mathbb{R}^{1+n}_+ = (0,\infty) \times \mathbb{R}^n$. Rappelons que pour $p \in (0,\infty)$, l'espace de tente T^p est défini comme l'ensemble des fonctions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ satisfaisant

$$\|f\|_{T^p} := \left(\int_{\mathbb{R}^n} \left(\iint_{|x-y|<t} |f(t,y)|^2 \frac{dt\,dy}{t^{1+n}} \right)^{p/2} dx \right)^{1/p} < \infty.$$

Dans le cas $p = \infty$, l'espace de tente T^∞ (de type Carleson) est défini comme l'ensemble des fonctions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$ satisfaisant

$$\|f\|_{T^\infty} := \sup_{(x,r)\in\mathbb{R}^{1+n}_+} \left(\iint_{\widehat{B(x,r)}} |f(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2} < \infty.$$

Pour $(r,x) \in \mathbb{R}^{1+n}_+$, on rappelle que l'objet

$$\widehat{B} := \left\{(t,y) \in \mathbb{R}^{1+n}_+ : B(y,t) \subset B(x,r)\right\}$$

est appelé la tente sur (la boule) $B(x,r)$ avec la hauteur r.

Posons $\mathbb{N}_+ = \{1,2,\cdots\}$.

Théorème 0.0.1. *Pour toutes $f \in T^p$, $0 < p < \infty$, et tous $l > 0$, il existe $C = C(n,p) > 0$ alors on peut trouver une famille des boules $\{B_i\}_{i\in\mathbb{N}_+}$ dans $\mathbb{R}^n$ et une décomposition de type Calderón-Zygmund*

$$f = g + \sum_{i\in\mathbb{N}_+} b_i,$$

avec la propriété $\operatorname{supp} b_i \subset \widehat{B_i}$, $\forall\, i \in \mathbb{N}_+$, *et des estimations*

$$\|g\|_{T^\infty} \le Cl,$$

$$\|b_i\|^p_{T^p} \le Cl^p |B_i|, \quad \forall\, i \in \mathbb{N}_+,$$

et

$$\sum_{i\in\mathbb{N}_+} |B_i| \le Cl^{-p} \|f\|^p_{T^p}.$$

De plus, les supports des b_i sont mutuellement disjoints.

Et puis, nous définissons pour $\lambda > 1$ et pour f mesurable

$$\mathscr{S}^*_\lambda(f)(x) := \left(\iint\limits_{\mathbb{R}^{1+n}_+} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} |f(t,y)|^2 \frac{dt\,dy}{t^{n+1}}\right)^{1/2}, \; x \in \mathbb{R}^n.$$

Utilisant ces dernières décompositions, on obtient le corollaire suivant.

Corollaire 0.0.2. *Soit $\lambda > 1$. Alors $\mathscr{S}^*_\lambda$ est borné de $T^{2/\lambda}$ dans $L^{2/\lambda,\infty}$.*

Ici, on rappelle que $L^{2/\lambda,\infty}$ est un espace de Lorentz (dans $\mathbb{R}^n$). Ce résultat est une généralisation d'une estimation dans la thèse de C. Fefferman [Fef70].

Chapitre 2. Nous proposons une théorie d'extrapolation de type Calderón-Zygmund pour les opérateurs sous-linéaires agissant sur l'échelle des espaces de tentes introduits par R. R. Coifman, Y. Meyer et E. M. Stein dans [CMS85]. Comme application, nous prouvons des estimations faibles étendant l'article [AKMP12].

L'ingrédient principal dans l'établissement de cette théorie d'extrapolation est 1) l'utilisation de certaines décompositions de type Calderón-Zygmund dans des espaces de tentes, qui sont obtenues dans chapitre 1, et dans l'application de cette théorie abstraite à la classe des opérateurs d'intégrale singulière sur les espaces de tentes qui a été considérée dans [AKMP12], 2) l'utilisation de certains plongements de type Hardy-Littlewood pour les fonctions des espaces de tentes.

- - - - - - - - - - - - Plus Précisément - - - - - - - - - - - -

Soit $m \in \mathbb{N}_+$ et $\beta \in \mathbb{R}$. Posons

$$\mathscr{A}(f)(x) := \left(\iint_{\mathbb{R}_+^{1+n}} \frac{\mathbf{1}_{B\left(x, t^{\frac{1}{m}}\right)}(y)}{t^{\frac{n}{m}}} |f(t,y)|^2 t^\beta \, dt dy \right)^{1/2}, \ x \in \mathbb{R}^n.$$

Nous définissons $T_\beta^{p,2,m}$, $0 < p < \infty$, par

$$\|f\|_{T_\beta^{p,2,m}} := \left\|\mathscr{A}(f)\right\|_{L^p} < \infty,$$

et définissons ${}^w T_\beta^{p,2,m}$, $0 < p < \infty$, par

$$\|f\|_{{}^w T_\beta^{p,2,m}} := \left\|\mathscr{A}(f)\right\|_{L^{p,\infty}} < \infty.$$

Soit $1 \leq r_1 \leq r_2 \leq \infty$. Posons $\triangle^c := \left\{(t,s) \in \mathbb{R}_+^2 \mid t \neq s\right\}$. Une famille d'opérateurs $\{K(t,s)\}_{(t,s)\in\triangle^c} \subset \mathscr{L}\left(L^2(\mathbb{R}^n)\right)$, satisfait à la décroissance $L^{r_1} - L^{r_2}$ hors-diagonale, avec homogénéité $m \in \mathbb{N}_+$ et ordre de décroissance $M > 0$, si

$$\|\mathbf{1}_F K(t,s) \mathbf{1}_E f\|_{L^{r_2}} \lesssim |t-s|^{-1-\frac{n}{m}(\frac{1}{r_1}-\frac{1}{r_2})} \left\langle 1 + \frac{\operatorname{dist}(E,F)^m}{|t-s|} \right\rangle^{-M} \|\mathbf{1}_E f\|_{L^{r_1}}$$

pour tous les ensembles de Borel $E, F \subset \mathbb{R}^n$, tous $(t,s) \in \triangle^c$ et toutes $f \in L^{r_1} \cap L^2$.

Soit $1 \leq q \leq 2$. On dit $T \in SIO^+_{m,q,M}$ si $T \in SIO^+$ et son noyau associé K satisfait à la décroissance $L^q - L^2$ hors-diagonale avec homogénéité m et ordre de décroissance M.

Théorème 0.0.3. *Soit $m \in \mathbb{N}_+$ et $\beta < 1$. Soit $T \in SIO^+_{m,q,M}$ avec $1 \leq q \leq 2$, $M > \frac{n}{2m}$ et soit $p_M < 1$ donné par $M = \frac{n}{m}\left(\frac{1}{p_M} - \frac{1}{2}\right)$. Soit q' l'exponent dual de q.*

(1) Si $q' \leq \frac{2n}{m(1-\beta)}$ ou de façon équivalente

$$\frac{n}{2m} \geq -\frac{\beta-1}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right),$$

alors T s'étend en un operator borné sur $T_\beta^{p,2,m}$ quand $\infty \geq p > p_c$, où

$$p_c = \frac{4n}{2n + m(1-\beta)q'} \geq 1.$$

(2) Si $q' > \frac{2n}{m(1-\beta)}$ ou de façon équivalente

$$\frac{n}{2m} < -\frac{\beta-1}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right),$$

alors T s'étend en un operator borné appliquant $T_\beta^{\widehat{p}_c,2,m}$ dans ${}^w T_\beta^{\widehat{p}_c,2,m}$, donc borné sur $T_\beta^{p,2,m}$ quand $\infty \geq p > \widehat{p}_c$, où $\widehat{p}_c = \max\left(p_M, \widetilde{p}_c\right)$ et

$$\widetilde{p}_c = \frac{2n}{\frac{2n}{q} + m(1-\beta)} < 1.$$

Donc (1) est dans [AKMP12] mais notre preuve est un peu différente. L'inégalité de type faible dans (2) est notre résultat. Nous remarquons que les espaces de tentes T^p, $0 < p < \infty$, peuvent être considérés comme des fonctions de $L^p(\mathbb{R}^n)$ à valeurs dans $L^2\left(\mathbb{R}_+^{1+n}\right)$. La théorie de type Calderón-Zygmund que nous avons établie dans $T^p\left(\mathbb{R}_+^{1+n}\right)$ diffère de celle dans $L^p(\mathbb{R}^n)$ à valeurs dans $L^2\left(\mathbb{R}_+^{1+n}\right)$.

0.2 Opérateurs de Régularité Maximale —— Stabilité de R-analyticité

Dans la deuxième partie, nous nous intéressons à **l'opérateur de régularité maximale** : soit $\left\{e^{-t\Lambda}\right\}_{t>0}$ un semi-groupe analytique, on pose

$$\mathbf{M}_\Lambda^+(F)_t = \int_0^t \Lambda e^{-(t-s)\Lambda} F_s \, ds \tag{0.0.1}$$

où la fonction F appartient à l'espace de tente T^p avec $0 < p \leq \infty$. Donc, $\mathbf{M}_\Lambda^+$ et un opérateur d'intégrale singulière dans la classe SIO^+.

On peut considérer trois types de générateurs analytiques pour Λ. Ils sont

$$\begin{cases} \text{1) Opérateurs elliptiques complexes} & L = -\mathrm{div} A\nabla \\ \text{2) Racines carrées d'opérateurs elliptiques complexes} & \sqrt{L} \\ \text{3) Opérateurs de Dirac perturbés d'ordre un} & |DB_0| \text{ et } |B_0 D|. \end{cases}$$

1) Pour des opérateurs elliptiques d'ordre deux, $L = -\mathrm{div}A\nabla$ dans $\mathbb{R}^n$, le semi-groupe de la chaleur a des estimations de décroissance $L^q - L^2$ hors-diagonale exponentielle. Avec des estimations de décroissance hors-diagonale assez rapide nous avons une décomposition efficace pour les opérateurs de régularité maximale et nous pouvons prouver pour $\mathbf{M}_L^+$ la bornitude satisfaisante dans les espaces de tentes.

2) La méthode de [AKMP12], qui utilise des estimations de décroissance hors-diagonale de type $L^q - L^2$, bénéficie évidemment de la croissance linéaire de l'ordre

$$M_{q2} := 1 + n\left(\frac{1}{q} - \frac{1}{2}\right)$$

en $\frac{1}{q}$, ainsi leur théorie est adaptée à $\mathbf{M}_{\sqrt{L}}^+$. Certaines adaptations (non incluses dans cette thèse) sont nécessaires pour prendre soin du cas quand q est proche de 2.

3) On va discuter le troisième type de générateurs dans la Partie III.

Chapitre 3. Nous donnons une condition suffisante pour la régularité maximale dans les espaces de tentes, c'est-à-dire, la bornitude dans les espaces de tentes des opérateurs de régularité maximale.

En particulier, pour un opérateur elliptique L de $2m$-ordre, à valeurs complexes, pas nécessairement sous forme de divergence, sur $\mathbb{R}^n$, avec m et n deux entiers plus grand que 1, nous montrons dans ce chapitre que les opérateurs L-associés de régularité maximale $\mathbf{M}_L^+$ s'étendent en des opérateurs bornés sur l'espace de tente parabolique $T^{p,2,2m}\left((0,\infty)\times\mathbb{R}^n\right)$ pour

$$(p_-)_* := \frac{np_-}{n+mp_-} < p \leq \infty,$$

avec $p_- \in [1,2)$ étant la borne inférieure de q, pour lequel le semi-groupe analytique complexe $\left\{e^{-zL}\right\}_{z\in S_\delta}$, $\delta\in(0,\pi/2)$, où

$$S_\delta = \left\{te^{i\theta} : t>0, |\arg\theta|<\delta\right\},$$

satisfait certaine décroissance $L^q\left(\mathbb{R}^n\right) - L^2\left(\mathbb{R}^n\right)$ hors diagonale d'ordre assez grande.

La partie intéressante réside dans la bornitude dans les espaces de tentes pour $p_- < p < 2$, et nos outils sont particulièrement conçus pour prendre soin du cas

$$(p_-)' \leq \frac{n}{m}, \text{ où } \frac{1}{p_-} + \frac{1}{(p_-)'} = 1.$$

Ceci complète ainsi la théorie d'extrapolation récente de [AMP12] et [AKMP12].

Nos hypothèses sur la décroissance $L^q\left(\mathbb{R}^n\right) - L^2\left(\mathbb{R}^n\right)$ hors diagonale du semi-groupe analytique sont plus faibles que celles requises par les critères d'extrapolation de Blunck-Kunstmann [BK02] et de Kunstmann [Kun08] de la R-analyticité du générateur analytique dans $L^p\left(\mathbb{R}^n\right)$ et de la régularité-$L^p\left(\mathbb{R}^n\right)$ maximale pour $q<p<2$.

- - - - - - - - - - - - Plus Précisément - - - - - - - - - - - -

Nous définissons l'espace de tente $T^{p,2,m}(dtdy)$, $0<p<\infty$, comme l'espace de toutes les fonctions $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ avec

$$\|F\|_{T^{p,2,m}(dtdy)} := \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{1+n}_+}\frac{\mathbf{1}_{B(x,t^{1/m})}(y)}{t^{n/m}}|F(t,y)|^2dtdy\right)^{p/2}dx\right)^{1/p} < \infty,$$

et définissons l'espace de tente $T^{\infty,2,m}(dtdy)$ comme l'espace de toutes les fonctions $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ avec

$$\|F\|_{T^{\infty,2,m}(dtdy)} := \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(r^{-n}\int_0^{r^m}\int_{B(x,r)}|F(t,y)|^2dtdy\right)^{1/2} < \infty.$$

Une famille d'opérateurs uniformément bornés dans $L^2(\mathbb{R}^n)$, $\{T(z)\}_{z\in A}$, $A\subset\mathbb{C}$, est satisfaire à la décroissance $L^q - L^r$ hors-diagonale, avec homogénéité m, si $\forall M>0$,

$$\left\|\mathbf{1}_{B_2}T(z)\mathbf{1}_{B_1}f\right\|_r \lesssim |z|^{-\frac{n}{m}\left(\frac{1}{q}-\frac{1}{r}\right)}\left\langle 1+\frac{\operatorname{dist}(B_1,B_2)^m}{|z|}\right\rangle^{-M}\left\|\mathbf{1}_{B_1}f\right\|_q$$

pour toutes les boules $B_1, B_2\subset\mathbb{R}^n$, tous $z\in A$ et toutes $f\in L^q(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Théorème 0.0.4. *Soit $1\le q<2$ et supposant $mq'\le 2n$ alors $q_*=\frac{2nq}{2n+mq}\ge 1$. Soit $\left\{e^{-t\Lambda}\right\}_{t>0}$ un semi-groupe analytique dans $L^2(\mathbb{R}^n)$.*

I) Supposons que la famille dérivée $\left\{t\Lambda e^{-t\Lambda}\right\}_{t>0}$ satisfait à la décroissance L^2-L^2 hors-diagonale avec homogénéité m. Supposons que $\left\{e^{-t\Lambda}\right\}_{t>0}$ satisfait à la décroissance L^q-L^2 hors-diagonale avec homogénéité m. Alors $\mathbf{M}_\Lambda^+$, originalement défini comme dans (0.0.1), s'étend en un opérateur borné sur $T^{p,2,m}(dt\,dy)$ pour $q_*<p\le\infty$.

II) Supposons que $\left\{e^{-z\Lambda}\right\}_{z\in S_\delta}$, pour un $0<\delta<\pi/2$, satisfait à la décroissance L^q-L^2 hors-diagonale avec homogénéité m. Alors $\mathbf{M}_\Lambda^+$, originalement défini comme dans (0.0.1), s'étend en un opérateur borné sur $T^{p,2,m}(dt\,dy)$ pour $q_*<p\le\infty$.

Ce théorème permet d'améliorer les résultats obtenus dans [AKMP12]. L'ingrédient pour la preuve permet d'améliorer le critère de la R-analyticité dans [BK02].

0.3 Calcul Opérationnel Holomorphe —— Analyse de/sur Moyennes de Whitney

Posons $N=(1+n)m$. Rappelons que l'opérateur de Dirac du premier ordre

$$D:=\begin{bmatrix}0 & \operatorname{div}_x\\ -\nabla_x & 0\end{bmatrix}$$

et la multiplication $B_0\in L^\infty\left(\mathbb{R}^n;\mathscr{L}\left(\mathbb{C}^N\right)\right)$. Le formalisme fondé par Auscher-Axelsson-McIntosh [AAM10], et puis, développé par Auscher-Axelsson [AA11] nous permet de transformer le système (d'ordre m) elliptique sous forme divergence d'ordre deux

$$\operatorname{div}_{t,x}A(t,x)\nabla_{t,x}u=0$$

en un système non-autonome du premier ordre

$$\partial_t F+DB_0F=D(\mathbf{E}F).$$

Ici, $A\in L^\infty\left(\mathbb{R}_+^{1+n};\mathscr{L}\left(\mathbb{C}^N\right)\right)$ vérifie une condition d'ellipticité, F est liée au gradient de u par $\begin{bmatrix}\nabla_x u\\ \partial_A u\end{bmatrix}$, et B est associé à A par une certaine relation algébrique. La perturbation $\mathbf{E}=B_0-B$ est bornée, et dans certains cas, vérifie une condition de Carleson.

Nous sommes donc intéressés par une famille d'opérateurs

$$\begin{cases} \mathbf{S}^{+,\varepsilon}_{DB_0}(\mathbf{E}F)_t = \int_0^t \eta^+_\varepsilon(t,s)e^{-(t-s)|DB_0|}\chi^+(DB_0)D(\mathbf{E}F)_s\,ds \\ \mathbf{S}^{-,\varepsilon}_{DB_0}(\mathbf{E}F)_t = \int_t^\infty \eta^-_\varepsilon(t,s)e^{-(s-t)|DB_0|}\chi^-(DB_0)D(\mathbf{E}F)_s\,ds \end{cases}$$

où $\eta^\pm_\varepsilon(t,s)$ se rapprochent des fonctions caractéristiques des intervalles $(0,t)$ et (t,∞) quand $\varepsilon \to 0$, et on pose

$$\mathbf{S}_{DB_0} = \lim_{\varepsilon\to 0}\left(\mathbf{S}^{+,\varepsilon}_{DB_0} - \mathbf{S}^{-,\varepsilon}_{DB_0}\right).$$

La définition rigoureuse de $\mathbf{S}_{DB_0}$ nécessite le **calcul opérationnel holomorphe**. En même temps, les projecteurs $\chi^\pm(DB_0)$ nécessite le **calcul fonctionnel holomorphe**.

Chapitre 4. Dans ce chapitre, nous introduisons une nouvelle échelle d'espaces de tentes qui couvre, les espaces de tentes de Coifman-Meyer-Stein et de Hofmann-Mayboroda-McIntosh, et quelques autres espaces de tentes considérés par Dahlberg, Kenig-Pipher et Auscher-Axelsson pour les systèmes elliptiques rugueux. Les factorisations au sein de nos espaces de tentes, avec des applications à l'interpolation complexe dans le cas Banachique ou quasi-Banachique, aux espaces de multiplicateurs ponctuels et à la dualité, sont établies. De cette façon, nous unifions et étendons les résultats correspondants obtenus par Coifman-Meyer-Stein, Cohn-Verbitsky et Hytönen-Rosén.

Nous ne rappelons pas les résultats de ce chapitre, qui nécessitent beaucoup de définitions. On remarque que ces factorisations (fortes) sont inspirées par [AB79] et [CV00]. Ce chapitre se trouve dans un article à paraître dans Math. Z. (2015).

Chapitre 5. Ce chapitre peut être plus ou moins considéré comme une continuation sous l'aspect d'analyse harmonique du mémoire récent [AS14] — P. Auscher et S. Stahlhut, *A priori estimates for boundary value elliptic problems via first order systems*, Mars 2014 — dans le cas des **coefficients t-dépendants** pour les systèmes elliptiques. Son but principal est d'extrapoler dans des espaces de tentes, des estimations de régularité maximale pondérées **a priori** obtenues par P. Auscher et A. Axelsson dans [AA11].

Par exemple, nous montrons dans ce chapitre que l'opérateur de régularité maximale associé à l'opérateur de Dirac perturbé B_0D, défini formellement par

$$F \mapsto \int_0^t B_0De^{-(t-s)|B_0D|}\chi^+(B_0D)F_s\,ds \\ - \int_t^\infty B_0De^{-(s-t)|B_0D|}\chi^-(B_0D)F_s\,ds,$$

s'étend en un opérateur borné sur une échelle d'espaces de tentes elliptiques pondérés $T^p_\beta\left(\mathbb{R}^{1+n}_+;\mathbb{C}^{(1+n)m}\right)$, où m est le nombre d'équations, pour

$$p_- < p < p_+ \quad \text{et} \quad \beta \in (-1,1),$$

et modulo une hypothèse sur l'action des $\chi^\pm(DB_0)$ sur certains espaces, pour

$$\max\left\{\frac{np_-}{n+\frac{1-\beta}{2}p_-},1\right\} < p < p_+ \quad \text{et} \quad \beta \in (-1,1),$$

où $\left(p_-,p_+\right)$ est l'intervalle des $p \in (1,\infty)$ pour lesquels B_0D admet un calcul fonctionnel holomorphe borné dans $L^p\left(\mathbb{R}^n;\mathbb{C}^{(1+n)m}\right)$. Rappelons que $\chi^\pm(B_0D)$ sont les projections spectraux de l'opérateur bisectoriel B_0D, et

$$|B_0D| = B_0D\left(\chi^+(B_0D)-\chi^-(B_0D)\right)$$

est le générateur du semi-groupe analytique $\left\{e^{-t|B_0D|}\right\}_{t>0}$ sur $L^2\left(\mathbb{R}^n;\mathbb{C}^{(1+n)m}\right)$. Dans le cas $\beta=-1$, nous montrons comment appliquer les résultats de dualité de [HR13] et du chapitre 4 pour obtenir des estimations similaires de régularité maximale.

Remarquons que l'opérateur

$$\operatorname{sgn}(B_0D) = \chi^+(B_0D)-\chi^-(B_0D),$$

étroitement lié au problème de la racine carrée de Kato résolu par P. Auscher et al. [AHL+02], provoque la principale difficulté dans l'analyse de l'opérateur de régularité maximale ci-dessus. Au niveau technique, nos arguments révèlent que la théorie L^2 des opérateurs de Dirac est suffisante dans l'estimation d'une partie singulière se rapportant à l'opérateur de régularité maximale. Les restrictions sur l'intervalle de p d'extrapolation proviennent d'une partie régulière. Plus précisément, pour la partie singulière, nous faisons l'usage de l'extrapolation dans les espaces de tentes (théorie L^2) de P. Auscher et al. dans [AMP12], et pour la partie régulière, nous faisons l'usage de l'extrapolation dans les espaces de tentes (théorie L^p) de P. Auscher et al. dans [AKMP12] et développons les techniques d'extrapolation conçues dans le chapitre 3.

Ces résultats de régularité maximale conique pondérés a priori pour les systèmes elliptiques d'ordre un, ont des répercussions sur certaines formules de Cauchy non-intégrales qui nous permettent de construire des solutions faibles pour les systèmes elliptiques t-dépendants. De cette façon nous étendons les résultats de A. Rosén [Ros14].

------------ Plus Précisément ------------

Pour $0<p<\infty$ et $\beta\in\mathbb{R}$, nous définissons T^p_β (analogue pondéré de T^p) comme l'espace de toutes les fonctions $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ avec

$$\|F\|_{T^p_\beta} := \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{1+n}_+}\frac{\mathbf{1}_{B(x,t)}(y)}{t^n}|F(t,y)|^2 t^\beta\, dt\, dy\right)^{p/2} dx\right)^{1/p} < \infty.$$

Soit $\alpha \geq 0$ et $\beta \in \mathbb{R}$. Nous définissons $T^{\infty,\alpha}_{\beta}$ par

$$\|F\|_{T^{\infty,\alpha}_{\beta}} := \sup_{(r,x)\in\mathbb{R}^{1+n}_{+}} \left(r^{-(n+2\alpha)} \iint_{(0,r)\times B(x,r)} |F(t,y)|^2 t^{\beta} \, dt\, dy \right)^{1/2} < \infty.$$

On a l'identification des espaces "dual" : pour $q \in (1,\infty)$

$$\left(T^{q}_{\beta}\right)' = T^{q'}_{-\beta},$$

et pour $q \in (0,1]$, $\alpha = n\left(\frac{1}{q} - 1\right)$

$$\left(T^{q}_{\beta}\right)' = T^{\infty,\alpha}_{-\beta},$$

avec la dualité donnée par

$$\langle F, G\rangle := \iint_{\mathbb{R}^{1+n}_{+}} F(t,y)\overline{G(t,y)}\, dt\, dy.$$

Cette dualité diffère de celle utilisée dans [CMS85].

On a des résultats de régularité maximale conique suivant.

Théorème 0.0.5. *Supposant* $\mathbf{E} \in L^{\infty}$.

(i) Pour $\beta \in (-1,1)$ *et* $p_- < p < p_+$ *on a*

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{T^{p}_{\beta}} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{T^{p}_{\beta}}.$$

(ii) Pour $\beta \in (-1,1)$ *et* $\widetilde{p}_- < \widetilde{p} < \widetilde{p}_+$ *on a*

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}.$$

Ici, dans (ii), $\left(\widetilde{p}_-, \widetilde{p}_+\right)$ *est l'intervalle des* $p \in (1,\infty)$ *pour lesquels* $B_0^* D$ *admet un calcul fonctionnel holomorphe borné dans* $L^{p}\left(\mathbb{R}^n; \mathbb{C}^{(1+n)m}\right)$.

Remarquons que $(p_-, p_+) = (1,\infty)$ pour $B_0 = I$.

Quand $\beta = \pm 1$, il faut supposer une condition de type Carleson sur $\mathbf{E}$, où on peut utiliser Chapitre 4 pour traiter la multiplication $\mathbf{E}F$, et on a des résultats similaires, dès que, on modifie l'espace où vivent F et $\mathbf{S}_{DB_0}(\mathbf{E}F)$. Donce les cas $\beta = \pm 1$ comptent une analyse de $\mathbf{S}_{DB_0} \circ \mathbf{E}$ sur des moyennes de Whitney. Sous certaines hypothèses concernant l'action sur T^{p} des projecteurs spectraux $\chi^{\pm}(DB_0)$, on peut améliorer le théorème ci-dessus pour p ou $\widetilde{p}$ hors de l'intervalle d'existence pour le calcul fonctionnel.

Ces résultats de régularité maximale conique sont motivés par la représentation des solutions de l'équation du premier ordre non-autonome sous la forme

$$F = \left(I - \mathbf{S}_{DB_0}\mathbf{E}\right)^{-1} e^{-t|DB_0|} h^{+} \tag{0.0.2}$$

où h^{+} est dans un espace de Hardy associé à DB_0 identifié à l'espace de Hardy associé à D pour les valeurs de p prises ici. Enfin nous remarquons que (0.0.2) est une intégrale de Cauchy, par rapport à la situation classique comme dans [Ros14, Example 1.5].

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Part I

SINGULAR INTEGRAL OPERATORS
—— CALDERÓN-ZYGMUND THEORY

1

Calderón-Zygmund decompositions in tent spaces and weak-type endpoint bounds for two quadratic functionals of Stein and Fefferman-Stein

Abstract

In this chapter, we prove some Calderón-Zygmund type decompositions for Coifman-Meyer-Stein tent space functions. These decompositions will be used in Chapter 2 in the study of certain singular integral operators on tent spaces. As application we give a unified proof for tent space generalizations of C. Fefferman's endpoint weak-type estimates for grand square functions and of C. Fefferman and Stein's endpoint weak-type estimates for box maximal functions.

Contents

1.1 Introduction

Let $\mathbb{R}^{1+n}_+ = \mathbb{R}_+ \times \mathbb{R}^n$, with $\mathbb{R}_+ = (0,\infty)$. For $t > 0$ and $y \in \mathbb{R}^n$, (t, y) denotes a point in $\mathbb{R}^{1+n}_+$. Let $B(x, t) \subset \mathbb{R}^n$ be the open ball which is centered at $x \in \mathbb{R}^n$ and has radius $t > 0$.

For $0 < p \leq \infty$, let $\|\cdot\|_p$ be the $L^p(\mathbb{R}^n)$ quasi-norm. Denote by $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ the collection of locally square integrable functions in $\mathbb{R}^{1+n}_+$. For $0 < p < \infty$ and $\alpha > 0$, we say an $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ function f belongs to the α-apertured tent space ${}^{\alpha}T^p_2$ if

$$\|f\|_{{}^{\alpha}T^p_2} := \left\|\mathscr{A}^{(\alpha)}(f)\right\|_p < \infty,$$

where

$$\mathscr{A}^{(\alpha)}(f)(x) := \left(\iint_{\mathbb{R}^{1+n}_+} \frac{1_{B(x,\alpha t)}(y)}{t^n} |f(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n. \tag{1.1.1}$$

Note that the scale of α-apertured tent space ${}^{\alpha}T^p_2$ has equivalent quasi-norms for different α, that is, for any $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ function f we have [1]

$$\|f\|_{{}^{\alpha}T^p_2} \simeq \|f\|_{{}^{\beta}T^p_2}, \quad 0 < p < \infty, \quad 0 < \alpha, \beta < \infty. \tag{1.1.2}$$

We omit the aperture parameter α in $\mathscr{A}^{(\alpha)}$ and ${}^{\alpha}T^p_2$ if $\alpha = 1$. Let

$$\widehat{\Omega} := \left\{(t, y) \in \mathbb{R}^{1+n}_+ \mid B(y, t) \subset \Omega\right\}$$

be the tent over the set $\Omega \subset \mathbb{R}^n$. Let $|E|$ be the volume of the set E in $\mathbb{R}^n$. Then we say an $L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ function f is in the tent space T^∞_2 if

$$\|f\|_{T^\infty_2} := \|\mathscr{C}(f)\|_\infty < \infty,$$

where

$$\mathscr{C}(f)(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \iint_{\widehat{B}} |f(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n. \tag{1.1.3}$$

Here the supremum is taken over all the balls in $\mathbb{R}^n$ which also contain x. Note that T^p_2, $0 < p \leq \infty$, is the scale of Coifman-Meyer-Stein tent spaces introduced in [CMS85].

In this chapter we give some Calderón-Zygmund type decompositions for tent space functions in T^p_2, $0 < p < \infty$. Let $\mathbb{N}^*$ be the set of integers not less than 1.

Theorem 1.1.1. *For any $f \in T^p_2$, $0 < p < \infty$, and any $l > 0$, there exists $C = C(n, p) > 0$ such that we can always find a family of balls $\{B_i\}_{i\in\mathbb{N}^*}$ in $\mathbb{R}^n$ and a Calderón-Zygmund type decomposition $f = g + \sum_{i\in\mathbb{N}^*} b_i$, with* $\operatorname{supp} b_i \subset \widehat{B_i}$, *such that*

$$\|g\|_{T^\infty_2} \leq Cl, \tag{1.1.4}$$

1. This change of aperture result can be found, for example, in [CMS85, Tor86, Aus11].

$$\|b_i\|_{T_2^p}^p \le Cl^p|B_i|, \tag{1.1.5}$$

and

$$\sum_{i\in\mathbb{N}^*} |B_i| \le Cl^{-p}\|f\|_{T_2^p}^p. \tag{1.1.6}$$

Moreover, the supports of b_i are mutually disjoint.

The number $l > 0$ involved in the above theorem is often called the *height* for the corresponding Calderón-Zygmund decomposition.

Let $\lambda > 1$. Define the grand square functional of Stein type as

$$\mathscr{S}_\lambda^*(f)(x) := \left(\iint_{\mathbb{R}_+^{1+n}} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} |f(t,y)|^2 \frac{dtdy}{t^{n+1}}\right)^{1/2}, x \in \mathbb{R}^n. \tag{1.1.7}$$

It can be easily verified that $\mathscr{S}_\lambda^*$ is $T_2^2 \to L^2$ bounded as $\lambda > 1$.

For f_0 in the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, and $P_t(x)$ being the Poisson kernel $t/\left(|x|^2+t^2\right)^{\frac{n+1}{2}}$, one has $f = t\nabla_{t,y}\left(f_0 * P_t\right) \in T_2^p$. Moreover, the grand square function

$$g_\lambda^*(f_0) := \mathscr{S}_\lambda^*(f),$$

which was first studied by A. Zygmund and E. Stein, satisfies the strong type estimate

$$\left\|g_\lambda^*(f_0)\right\|_p = \left\|\mathscr{S}_\lambda^*\left(t\nabla_{t,y}\left(f_0 * P_t\right)\right)\right\|_p \lesssim \left\|\mathscr{A}\left(t\nabla_{t,y}\left(f_0 * P_t\right)\right)\right\|_p \tag{1.1.8}$$

if $\lambda > \max(1, 2/p)$ (see [Ste70, Theorem 2, p. 91]). This result has a tent space generalization: for $0 < p < \infty$ and f locally square integrable in $\mathbb{R}_+^{1+n}$, one has

$$\left\|\mathscr{S}_\lambda^*(f)\right\|_p \lesssim \|\mathscr{A}(f)\|_p \tag{1.1.9}$$

if $\lambda > \max(1, 2/p)$ (see [Aus11]). The endpoint case of (1.1.9) for $\lambda > 1$ and $p = 2/\lambda \in (0,2)$ corresponds to a tent space generalization of the endpoint case of (1.1.8) for g_λ^*, the latter proved in C. Fefferman's thesis [Fef69] (see [Fef70, Theorem 1]).

For $0 < p < \infty$, let $L^{p,\infty}$ be the Lorentz space in $\mathbb{R}^n$. As application of Theorem 1.1.1, our first result is the following weak-type estimate for the quadratic functional $\mathscr{S}_\lambda^*$.

Corollary 1.1.2. *Let $\lambda > 1$. Then $\mathscr{S}_\lambda^*$ is $T_2^{2/\lambda} \to L^{2/\lambda,\infty}$ bounded.*

Corollary 1.1.2 has a natural companion. Let $\lambda > 1$. Define the box maximal functional of Fefferman-Stein type (see [FS72, Lemma 8 and Lemma 9])

$$\mathscr{C}_\lambda^*(f)(x) := \sup_{r>0}\left(\frac{1}{|B(x,r)|^\lambda}\iint_{(0,r)\times B(x,r)} t^{\lambda n-n}|f(t,y)|^2\frac{dtdy}{t}\right)^{1/2}, x \in \mathbb{R}^n. \tag{1.1.10}$$

Our second result is the following weak-type estimate for $\mathscr{C}_\lambda^*$.

Corollary 1.1.3. *Let $\lambda > 1$. Then $\mathscr{C}_\lambda^*$ is $T_2^{2/\lambda} \to L^{2/\lambda,\infty}$ bounded.*

It can be verified that $\mathscr{C}_\lambda^*$ is $T_2^2 \to L^{2,\infty}$ bounded as $\lambda > 1$. This uses the relation

$$\mathscr{C}_\lambda^*(f)(x) \lesssim \mathscr{C}(f)(x), \quad \forall\, x \in \mathbb{R}^n,$$

together with the weak-type estimate $\mathscr{C} : T_2^2 \to L^{2,\infty}$ which was given in [CMS85, Theorem 3, (b)] as a consequence of maximal theorem.

Remark 1.1.4. The weak-type estimates in Corollary 1.1.2 and Corollary 1.1.3 are not new. Corollary 1.1.2 and the corresponding weighted norm inequalities were first proved in [AS77] [2] by using an integration lemma over the cones as in [CMS85, Lemma 2], which requires certain geometric properties of $\mathbb{R}^n$. Our proof is direct and has the possible extension to rough geometry. Corollary 1.1.3 was implied by some pointwise estimates in [CW83] (see also [Bar79] and [Tor79]) proved via Carleson measures.

Remark 1.1.5. Our Calderón-Zygmund decompositions will be used in Chapter 2 in establishing weak-type estimates for singular integral operators on tent spaces. Use of the functional $\mathscr{C}_\lambda^*$ in operator theory on tent spaces will be given somewhere else.

In proving the above corollaries, we shall also need the following L^2 estimate on the bad functions $\{b_i\}$ which arise from the Calderón-Zygmund decomposition.

Lemma 1.1.6. *Given any $f \in T_2^p$, with $0 < p < \infty$. Let $f = g + \sum_{i\in\mathbb{N}^*} b_i$ be the Calderón-Zygmund decomposition associated to the height $l > 0$ as in Theorem 1.1.1.*

If $0 < p < 2$ and $\lambda = 2/p > 1$, we have the fractional integral estimate

$$\iint_{\widehat{B_i}} t^{\lambda n-n} |b_i(t,y)|^2 \frac{dt\,dy}{t} \lesssim \|b_i\|_{T_2^p}^2 \lesssim l^2 |B_i|^{2/p}. \tag{1.1.11}$$

Here $\{B_i\}_{i\in\mathbb{N}^*}$ is the family of balls found in Theorem 1.1.1.

Note that the proof of the first estimate in (1.1.11) is essentially a general embedding estimate $T_2^p(\mathbb{R}_+^{1+n}) \hookrightarrow L^2(\mathbb{R}_+^{1+n}, t^{\lambda n-n-1}dt\,dy)$, with $0 < p < 2$. Thus the statement of this lemma can be more general, but this information will not be used in this chapter. In this regard, see Section 2.2.3 of next chapter for related tent space embedding results.

1.2 Proof of Theorem 1.1.1

Let $\mathbf{M}$ be the maximal function in $\mathbb{R}^n$, that is,

$$\mathbf{M}(h)(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |h|, \quad x \in \mathbb{R}^n.$$

2. We thank Professor Pascal Auscher for pointing out this reference.

Fix $\alpha > 7$. Let

$$\Omega_l = \left\{x \in \mathbb{R}^n : \mathbf{M}\left(\mathscr{A}^{(\alpha)}(f)^p\right)(x) > l^p\right\},$$

thus by maximal theorem and the lower semi-continuity of $\mathbf{M}\left(\mathscr{A}^{(\alpha)}(f)^p\right)(x)$, we know that Ω_l is an open set with finite measure. Denote by $\Omega_l = \cup_{i\in\mathbb{N}^*} Q_i$ the Whitney decomposition of Ω_l into cubes, and by $B_i = cQ_i$ the ball with the same center as Q_i and radius c times the diameter of Q_i. We choose c sufficiently large such that

$$\widehat{B_i} \supset \Delta_i := \widehat{\Omega_l} \cap (Q_i \times (0,\infty))$$

uniformly in $i \in \mathbb{N}^*$. Then we let

$$b_i = f|_{\Delta_i} \ \text{ and } \ g = f - \sum_{i\in\mathbb{N}^*} b_i.$$

We claim that this yields the desired decomposition.

Since $\alpha > 7$, we claim that for any $x \in Q_i$, there exists $x_i \in \complement(\Omega_l)$, the complement of Ω in $\mathbb{R}^n$, such that

$$\mathscr{A}(g)(x) \le \mathscr{A}^{(\alpha)}(g)(x_i).$$

In fact, we can simply select x_i to be an arbitrary point in the set $5Q_i \cap \complement(\Omega_l)$. Recall that in the general Whitney decomposition arguments (see [Ste70, Theorem 1, p. 167] for example) one has

$$\operatorname{diam}(Q_i) \le \operatorname{dist}\left(Q_i, \complement(\Omega_l)\right) \le 4\operatorname{diam}(Q_i). \tag{1.2.1}$$

Hence $5Q_i \cap \complement(\Omega_l)$ is non-empty. With $r_i = \operatorname{diam}(Q_i)/2$, we also have

$$\operatorname{dist}(x_i, Q_i) \le 4r_i \ \text{ and } \ \operatorname{dist}\left(Q_i, \complement(\Omega_l)\right) \ge 2r_i.$$

From geometrical observations, to meet

$$\sup_{x\in Q_i} \mathscr{A}(g)(x) \le \inf_{x_i \in 4Q_i \cap \complement(\Omega_l)} \mathscr{A}^{(\alpha)}(g)(x_i),$$

it suffices to take

$$\begin{aligned} \alpha &> \frac{\operatorname{dist}(x_i, Q_i) + \operatorname{diam}(Q_i) + \frac{1}{2}\operatorname{dist}\left(Q_i, \complement(\Omega_l)\right)}{\frac{1}{2}\operatorname{dist}\left(Q_i, \complement(\Omega_l)\right)} \\ &= \frac{4r_i + 2r_i}{r_i} + 1 = 7. \end{aligned} \tag{1.2.2}$$

This proves the claim. Note that this claim is uniform in $i \in \mathbb{N}^*$.

Now for the "good" part g, with $C = C(n,p)$ different at each step, we have

$$\begin{aligned} \|\mathscr{C}(g)\|_{L^\infty} &\le C\|\mathscr{A}(g)\|_{L^\infty} \\ &\le C\left\|\mathscr{A}(g)|_{\Omega_l}\right\|_{L^\infty} + C\left\|\mathscr{A}(g)|_{\complement(\Omega_l)}\right\|_{L^\infty} \\ &\le C\left\|\mathscr{A}^{(\alpha)}(f)|_{\complement(\Omega_l)}\right\|_{L^\infty} + C\left\|\mathscr{A}(f)|_{\complement(\Omega_l)}\right\|_{L^\infty} \\ &\le C\left\|\mathscr{A}^{(\alpha)}(f)|_{\complement(\Omega_l)}\right\|_{L^\infty} \le C\lambda. \end{aligned}$$

Here we mainly used in order, in the first inequality the endpoint comparison of $\mathscr{A}$ and $\mathscr{C}$ at $p=\infty$ (see [CMS85, Theorem 3 (b)]), in the third inequality the claim just proved

$$\mathscr{A}(g)(x) \leq \mathscr{A}^{(\alpha)}(g)(x_i) \leq \mathscr{A}^{(\alpha)}(f)(x_i), \quad \forall x \in Q_i$$

and the construction $f|_{\complement(\widehat{\Omega_l})} = g$, in the fourth inequality the geometrical fact $\alpha > 1$, and in the fifth inequality the Lebesgue differentiation theorem applied to $\mathscr{A}^{(\alpha)}(f)^p$.

By similar geometrical observations as in (1.2.2), there exists $k = k(n,c) \geq 5$ such that for any $i \in \mathbb{N}^*$, we have the α-aperture tent $\widehat{(kQ_i)^\alpha} \supset \widehat{B_i}$, where

$$\widehat{\Omega^\alpha} := \left\{(t,y) \in \mathbb{R}^{1+n}_+ \mid B(y,\alpha t) \subset \Omega\right\}.$$

Now for the "bad" part b, first we know that $\operatorname{supp} b_i \subset \widehat{B_i}$. With $C = C(n,p,c)$ different at each step, we can estimate

$$\begin{aligned} \|b_i\|_{T^p_2} &\leq \left\| f|_{\widehat{B_i}} \right\|_{T^p_2} \\ &\leq C \left\| \mathscr{A}^{(\alpha)}\left(f|_{\widehat{B_i}}\right) \right\|_p \\ &\leq C \left\| \mathscr{A}^{(\alpha)}(f)|_{kQ_i} \right\|_p \leq C\lambda |B_i|^{1/p}. \end{aligned} \tag{1.2.3}$$

Here we mainly used in order, $\alpha > 1$, $\widehat{(kQ_i)^\alpha} \supset \widehat{B_i}$, $k \geq 5$ and the existence of $x_i \in kQ_i \cap \complement(\Omega_l)$, and the construction of Ω_l from maximal function.

Moreover, by maximal theorem

$$\begin{aligned} \sum_{i\in\mathbb{N}^*} |B_i| &\leq C \sum_{i\in\mathbb{N}^*} |Q_i| \\ &= C|\Omega_l| \\ &\leq C l^{-p} \|f\|^p_{{}^\alpha T^p_2} \leq C l^{-p} \|f\|^p_{T^p_2}, \end{aligned} \tag{1.2.4}$$

where $C = C(n,p,c)$. The last estimate used (1.1.2).

Finally, using the fact that the Whitney cubes $\{Q_i\}$ are mutually disjoint (again see [Ste70, Theorem 1, p. 167]), we see that the supports of b_i are mutually disjoint.

1.3 Proofs of Corollaries 1.1.2 and 1.1.3

Proof of Corollary 1.1.2. Recall that we have the $T^2_2 \to L^2$ boundedness of $\mathscr{S}^*_\lambda$ when $\lambda > 1$. By density of $T^{2/\lambda}_2 \cap T^2_2$ in $T^{2/\lambda}_2$ it suffices to show for any $f \in T^{2/\lambda}_2 \cap T^2_2$

$$\left|\left\{x \in \mathbb{R}^n \middle| \mathscr{S}^*_\lambda(f)(x) > l\right\}\right| \lesssim \frac{1}{l^{2/\lambda}} \int_{\mathbb{R}^n} \mathscr{S}^*_\lambda(f)^{2/\lambda}(x)\,dx, \qquad \forall\, l > 0.$$

Let $f = g + \sum_{i\in\mathbb{N}^*} b_i$ be the Calderón-Zygmund decomposition associated to the height $l > 0$, the Whitney cubes $\{Q_i\}_{i\in\mathbb{N}^*}$ and the balls $\{B_i\}_{i\in\mathbb{N}^*}$ as in Theorem 1.1.1, such that $g = f|_{\complement(\widehat{\Omega})}$ with $\Omega = \cup_i Q_i$, and $b_i = f|_{\triangle_i}$ with $\triangle_i = (Q_i \times (0,\infty)) \cap \widehat{\Omega} \subset \widehat{B_i}$.

By sublinearity of the quadratic functional $\mathscr{S}_\lambda^*$

$$\mathscr{S}_\lambda^*(f)(x) \le \mathscr{S}_\lambda^*(g)(x) + \mathscr{S}_\lambda^*(f-g)(x) := G_1(x) + G_2(x), \quad \forall\, x \in \mathbb{R}^n,$$

then it reduces to check that G_k $(k = 1,2)$ is in $L^{2/\lambda,\infty}$.

By $T_2^2 \to L^2$ boundedness of $\mathscr{S}_\lambda^*$, we have

$$\left|\left\{x \in \mathbb{R}^n \middle| G_1(x) > l/2\right\}\right| \lesssim l^{-2} \|\mathscr{S}_\lambda^*(g)\|_{T_2^2}^2 \lesssim l^{-2} \|g\|_{T_2^2}^2.$$

By the interpolation control for g from the Calderón-Zygmund decomposition,

$$l^{-2}\|g\|_{T_2^2}^2 \lesssim l^{-2/\lambda} \|f\|_{T_2^{2/\lambda}}^{2/\lambda}.$$

This shows that $G_1 \in L^{2/\lambda,\infty}$.

By the property of the Calderón-Zygmund decomposition, we have for G_2

$$\left|\left\{x \in \mathbb{R}^n \middle| G_2(x) > l/2\right\}\right| \lesssim l^{-2/\lambda} \|f\|_{T_2^{2/\lambda}}^{2/\lambda} + \left|\left\{x \in \mathbb{R}^n \setminus \cup_i 4B_i \middle| G_2(x) > l/2\right\}\right|.$$

Let $\Omega^* = \cup_i 4B_i$, thus for any $x \in \mathbb{R}^n \setminus \Omega^*$, we have $|x - y_i| \sim |x - y|$, where y_i denotes the center of Q_i and B_i, and y is any point in Q_i. Therefore, for any $x \in \mathbb{R}^n \setminus \Omega^*$

$$\begin{aligned}
G_2^2(x) &= \iint_{\widehat{\Omega}} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} \left|\sum_{i\in\mathbb{N}^*} b_i(t,y)\right|^2 \frac{dtdy}{t^{n+1}} \\
&= \sum_{i\in\mathbb{N}^*} \iint_{\triangle_i} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} |b_i(t,y)|^2 \frac{dtdy}{t^{n+1}} \\
&\lesssim \sum_{i\in\mathbb{N}^*} \frac{1}{|x-y_i|^{\lambda n}} \iint_{\triangle_i} t^{\lambda n - n} |b_i(t,y)|^2 \frac{dtdy}{t} \\
&\lesssim \sum_{i\in\mathbb{N}^*} \frac{\|b_i\|_{T_2^{2/\lambda}}^2}{|x-y_i|^{\lambda n}} \lesssim \sum_{i\in\mathbb{N}^*} \frac{l^2 |B_i|^\lambda}{|x-y_i|^{\lambda n}},
\end{aligned}$$

where we used Lemma 1.1.6 in the last two estimates. In the second equality above, we also used the fact the supports of the bad functions $\{b_i\}$ are mutually disjoint, which is guaranteed by the Calderón-Zygmund decomposition.

Then it remains to show

$$\left|\left\{x \in \mathbb{R}^n \setminus \Omega^* \middle| H(x) > 1\right\}\right| \lesssim l^{-2/\lambda} \|f\|_{T_2^{2/\lambda}}^{2/\lambda},$$

with

$$H(x) := \sum_{i\in\mathbb{N}^*} \frac{|B_i|^\lambda}{|x-y_i|^{\lambda n}}.$$

However, by Tchebitchev inequality for $H(x)$ restricted to $\mathbb{R}^n \setminus \Omega^*$, we have

$$\left|\left\{x \in \mathbb{R}^n \setminus \Omega^* \middle| H(x) > 1\right\}\right| \le \int_{\mathbb{R}^n \setminus \Omega^*} H(x)dx$$

$$= \sum_{i \in \mathbb{N}^*} |B_i|^\lambda \int_{\mathbb{R}^n \setminus \Omega^*} \frac{dx}{|x - y_i|^{\lambda n}}$$
$$\leq \sum_{i \in \mathbb{N}^*} |B_i|^\lambda \int_{\mathbb{R}^n \setminus B_i} \frac{dx}{|x - y_i|^{\lambda n}} \lesssim \sum_{i \in \mathbb{N}^*} |B_i|.$$

Thus, with the property (1.1.6) from the Calderón-Zygmund decomposition, we finish the proof that $G_2 \in L^{2/\lambda,\infty}$.

The proof for Corollary 1.1.2 can be concluded by invoking the Marcinkiewicz interpolation theorem. □

Remark 1.3.1. It would be interesting to know if one can obtain for $\lambda > 1$ the Lorentz type estimate $\mathscr{S}_\lambda^* : T_2^{2/\lambda} \to L^{2/\lambda,2}$. The motivation of this question comes from [ST01].

Proof of Corollary 1.1.3. Recall that we have the $T_2^2 \to L^{2,\infty}$ boundedness of $\mathscr{C}_\lambda^*$ when $\lambda > 1$. We examine the preceding arguments carried out for $\mathscr{S}_\lambda^*$, and we notice that in estimating the new version of G_2^2, which we can write as

$$G_2^2(x) = \sum_{i \in \mathbb{N}^*} \sup_{r>0} \frac{1}{r^{\lambda n}} \iint_{\triangle_i \cap \{B(x,r) \times (0,r)\}} t^{\lambda n - n} |b_i(t,y)|^2 \frac{dt dy}{t},$$

we always have $r \geq C|x - y_i|$ in each summand since for $x \in \mathbb{R}^n \setminus \Omega^*$ and $y \in B(x,r)$, $|x - y_i| \sim |x - y|$ and $|x - y| \leq r$. Hence

$$G_2^2(x) \lesssim \sum_{i \in \mathbb{N}^*} \sup_{r>0} \frac{1}{|x - y_i|^{\lambda n}} \iint_{\triangle_i \cap \{B(x,r) \times (0,r)\}} t^{\lambda n - n} |b_i(t,y)|^2 \frac{dt dy}{t}, \quad x \in \mathbb{R}^n \setminus \Omega^*.$$

Furthermore, removing the region $B(x,r) \times (0,r)$ in the integrals, then

$$G_2^2(x) \lesssim \sum_{i \in \mathbb{N}^*} \frac{1}{|x - y_i|^{\lambda n}} \iint_{\triangle_i} t^{\lambda n - n} |b_i(t,y)|^2 \frac{dt dy}{t}$$
$$\lesssim \sum_{i \in \mathbb{N}^*} \frac{l^2 |B_i|^\lambda}{|x - y_i|^{\lambda n}}, \quad x \in \mathbb{R}^n \setminus \Omega^*.$$

Note that this goes back to the step in the above proof for $\mathscr{S}_\lambda^*$.

The other arguments remain unchanged. □

Remark 1.3.2. In the classical setting for harmonic extensions, namely $f = t\nabla_{t,y}\left(f_0 * P_t\right)$, both corollaries were proved in [MW74] exploiting the pointwise relation between $\mathscr{S}_\lambda^*$ and $\mathscr{C}_\lambda^*$. Such relations are not true for general tent space functions. Here we give a unified proof of Corollary 1.1.2 and Corollary 1.1.3 through our Calderón-Zygmund type decompositions in tent spaces, and this approach is close to the original spirits of [Fef70] and [FS72]. More precisely, see p. 20-21 of [Fef70] and p. 181-182 of [FS72]. Our arguments on G_2 in the above proofs reveal that, modulo the good function $g \in T_2^\infty$ in the Calderón-Zygmund type decomposition $f = g + b$, the quadratic functionals $\mathscr{S}_\lambda^*(b)(x)$ and $\mathscr{C}_\lambda^*(b)(x)$ have comparable upper bounds when the cone with vertex x doesn't intersect with the support of the bad function b.

1.4 Proof of Lemma 1.1.6

Observe that the second inequality in the claim (1.1.11) rewrites (1.1.5) in the Calderón-Zygmund decompositions. It suffices to prove the first inequality.

First, by a straightforward extension [3] of [CMS85, Theorem 1 (c)], the bad function $b_i \in T_2^p$, $0<p<2$, admits an atomic decomposition, say

$$b_i = \sum_j \lambda_{ij} b_{ij},$$

where the atom b_{ij} is supported in the tent $\widehat{B_{ij}}$ over some ball $B_{ij} \subset \mathbb{R}^n$ and b_{ij} also satisfies the size requirement

$$\|b_{ij}\|_{L^2(t^{-1}dtdy)} \le |B_{ij}|^{1-\frac{2}{p}}.$$

Moreover, the coefficients $\left\{\lambda_{ij}\right\}_j \in l^p$ satisfies

$$\left\|\left\{\lambda_{ij}\right\}_j\right\|_{l^p} \lesssim \|b_i\|_{T_2^p}. \tag{1.4.1}$$

We point out that the reverse $\|b_i\|_{T_2^p} \lesssim \left\|\left\{\lambda_{ij}\right\}_j\right\|_{l^p}$ is valid only for $p \le 1$.

Then we note that the decomposition equality $b_i = \sum_j \lambda_{ij} b_{ij}$ holds in pointwise sense. This follows by inspection of the proof of [CMS85, Theorem 1 (c)] (see also [Rus07] for more precise arguments).

Next it suffices to prove that the fractional integral estimate as in (1.1.11) holds uniformly on the atoms b_{ij}. Now we explain this in detail.

Note that $t \le r_{B_{ij}}$, the radius of the ball B_{ij}, and $\lambda n - n > 0$. The verification of the fractional integral estimate on atoms is as follows

$$\iint_{\widehat{B_{ij}}} t^{\lambda n-n} |b_{ij}(t,y)|^2 \frac{dtdy}{t} \lesssim \left(r_{B_{ij}}\right)^{\lambda n-n} |B_{ij}|^{1-\frac{2}{p}} \lesssim 1.$$

Hence, using that b_{ij} have disjoint support, we get

$$\begin{aligned}
\iint_{\triangle_i} t^{\lambda n-n} |b_i(t,y)|^2 \frac{dtdy}{t} &= \iint_{\triangle_i} t^{\lambda n-n} \left|\sum_j \lambda_{ij} b_{ij}(t,y)\right|^2 \frac{dtdy}{t} \\
&\le \sum_j \lambda_{ij}^2 \iint_{\widehat{B_{ij}}} t^{\lambda n-n} \left|b_{ij}(t,y)\right|^2 \frac{dtdy}{t} \\
&\lesssim \left\|\left\{\lambda_{ij}^2\right\}_j\right\|_{l^1}.
\end{aligned}$$

Since $p/2 = 1/\lambda < 1$

$$\left\|\left\{\lambda_{ij}^2\right\}_j\right\|_{l^1} \le \left\|\left\{\lambda_{ij}^2\right\}_j\right\|_{l^{p/2}} = \left\|\left\{\lambda_{ij}\right\}_j\right\|_{l^p}^2.$$

Combining this with the coefficient estimate (1.4.1), we finish the proof of this lemma.

3. See for example [Var07, p. 52] for such a precise statement. Be aware that the setup of [Var07] is quite different from that of [CMS85].

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2

Singular integral operators on tent spaces: a Calderón-Zygmund theory and weak-type endpoint estimates

Abstract

We propose a Calderón-Zygmund type extrapolation theory for sublinear operators acting on tent spaces introduced by R. R. Coifman, Y. Meyer and E. M. Stein in [CMS85]. As an application we prove endpoint weak-type estimates for the article (referred as [AKMP12])– P. Auscher, C. Kriegler, S. Monniaux and P. Portal, *Singular integral operators on tent spaces*, J. Evol. Equ. **12** (2012), 741–765.

The main ingredient in establishing this extrapolation theory is the use of some Calderón-Zygmund type decompositions in tent spaces, which are obtained in Chapter 1 of this thesis, and in applying this abstract theory to the class of singular integral operators on tent spaces as considered in [AKMP12], is the use of certain Hardy-Littlewood embeddings for tent space functions.

Contents

2.1 Introduction

The aim of this chapter is three-fold. First of all, we propose an abstract Calderón-Zygmund extrapolation theory for sublinear operators acting on tent spaces. Then we apply this extrapolation method to prove endpoint weak-type estimates for the article [AKMP12] – P. Auscher, C. Kriegler, S. Monniaux and P. Portal, *Singular integral operators on tent spaces*, J. Evol. Equ. **12** (2012), 741–765. In applying this abstract theory to singular integral operators on tent spaces as considered in [AKMP12], certain estimates (which we call *Hardy-Littlewood embeddings*) for tent space functions are indispensable. Our third contribution is then an extensive study on such embedding estimates.

First we review the notations and definitions employed in [AKMP12].

Let $m \in \mathbb{N}_+$, the integers not less than 1, and let $\beta \in \mathbb{R}$, the real numbers. Let $\mathbb{R}^{1+n}_+ = \mathbb{R}_+ \times \mathbb{R}^n$, with $\mathbb{R}_+ = (0,\infty)$. Define the functionals

$$\mathscr{A}(f)(x) := \left(\iint\limits_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B\left(x,t^{\frac{1}{m}}\right)}(y)}{t^{\frac{n}{m}}} |f(t,y)|^2 t^\beta \, dt\, dy \right)^{1/2}, \ x \in \mathbb{R}^n, \tag{2.1.1}$$

$$\mathscr{C}(f)(x) := \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \left(r^{-n} \int_0^{r^m} \int_{B(x,r)} |f(t,y)|^2 t^\beta \, dt\, dy \right)^{1/2}, \ x \in \mathbb{R}^n. \tag{2.1.2}$$

Here, $B(x,r)$ is the ball in $\mathbb{R}^n$ with center x and radius r. Note that there are implicit parameters m and β in the notations $\mathscr{A}$ and $\mathscr{C}$.

Denote by $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}\right)$ the collection of all locally square integrable complex-valued functions in $\mathbb{R}^{1+n}_+$. Denote by $\|\cdot\|_{L^p}$ the quasi-norm of the Lebesgue space $L^p = L^p(\mathbb{R}^n;\mathbb{C})$, $0 < p \le \infty$.

Definition 2.1.1. Define the tent space $T^{p,2,m}_\beta$, $0 < p < \infty$, as the space of all L^2_{loc} functions such that

$$\|f\|_{T^{p,2,m}_\beta} := \left\|\mathscr{A}(f)\right\|_{L^p} < \infty. \tag{2.1.3}$$

Define the Lorentz type tent space ${}^wT^{p,2,m}_\beta$, $0 < p < \infty$, as the space of all L^2_{loc} functions such that

$$\|f\|_{{}^wT^{p,2,m}_\beta} := \left\|\mathscr{A}(f)\right\|_{L^{p,\infty}} < \infty. \tag{2.1.4}$$

Here $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^n;\mathbb{C})$ is the usual Lorentz space in $\mathbb{R}^n$.

Define $T^{\infty,2,m}_\beta$ as the space of all L^2_{loc} functions such that

$$\|f\|_{T^{\infty,2,m}_\beta} := \left\|\mathscr{C}(f)\right\|_{L^\infty} < \infty. \tag{2.1.5}$$

The classical tent spaces, introduced by R. R. Coifman, Y. Meyer and E. M. Stein in [CMS85], corresponds to the scale $T^{p,2,m}_{\beta}$ when $m = 1$ and $\beta = -1$, for which we simply use the notation $T^{p,2}$, $0 < p \le \infty$. Be aware that the notation differs from Chapter 1. Similarly we write ${}^{w}T^{p,2,1}_{-1}$ as ${}^{w}T^{p,2}$, $0 < p < \infty$.

Note that the additional parameter 2 in $T^{p,2,m}_{\beta}$ designates the quadratic form in $\mathscr{A}$. Here we do not define the general case when 2 is replaced by q with $1 < q < \infty$. But we will study the extreme case when $q = \infty$ in Subsection 2.2.3 below.

Remark 2.1.2. The map $\iota \colon T^{p,2,m}_{\beta} \to T^{p,2}$, $0 < p \le \infty$, defined by

$$\iota(f)(t,y) := \sqrt{m}\, t^{\frac{m(1+\beta)}{2}} f(t^m, y)$$

is an isometry. The same applies to the scale ${}^{w}T^{p,2,m}_{\beta}$, $0 < p < \infty$.

Definition 2.1.3. We say $T \in SIO^{+}$ if T is bounded on $L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^n)\right)$, and if there exists a strongly measurable family of operator-valued kernels $K = \{K(t,s)\}_{\infty > t > s > 0}$ such that

$$\|K(t,s)\|_{\mathscr{B}(L^2(\mathbb{R}^n))} \le \frac{C}{t-s}$$

and

$$T(f)(t) = \int_0^t K(t,s) f(s)\, ds \tag{2.1.6}$$

for all $f \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^n)\right)$ with bounded support in $\mathbb{R}_+$ and almost all $t \in \mathbb{R}_+$ not in the support of f. Here SIO is Singular Integral Operator for short, and the sign "+" stands for $t > s$ in the kernel $K(t,s)$.

The above representation (2.1.6) of $T(f)$ is a Bochner integral and the equality holds in $L^2(\mathbb{R}^n)$. It is clearly equivalent to

$$\langle T(f), g\rangle = \iint_{s<t} \langle K(t,s) f(s), g(t)\rangle\, ds\, dt$$

if $f, g \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^n)\right)$ have bounded disjoint support. The inner product on the left is the canonical one in $L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^n)\right)$, and on the right the canonical one in $L^2(\mathbb{R}^n)$.

Denote the time off-diagonal by $\triangle^c := \left\{(t,s) \in \mathbb{R}^2_+ \mid t \ne s\right\}$. For $m \in \mathbb{N}_+$, let $\langle a/b\rangle_m := 1 + a^m/b$. For two Borel sets $E, F \subset \mathbb{R}^n$, let $\operatorname{dist}(E,F) := \inf\{|x-y| \mid x \in E, y \in F\}$.

Singular integral operators in Definition 2.1.3 have operator-valued kernels. As pioneered in [AMR08, HvNP08] and later in [AMP12, AKMP12], the following measure of decay on such kernels will be central to extrapolation problems on tent spaces.

Definition 2.1.4. Let $1 \le r_1 \le r_2 \le \infty$. An off-diagonal (in time) operator-valued kernel $\{K(t,s)\}_{(t,s)\in\triangle^c} \subset \mathscr{B}\left(L^2(\mathbb{R}^n)\right)$, is said to satisfy the $L^{r_1} - L^{r_2}$ off-diagonal (in space) decay, with homogeneity $m \in \mathbb{N}_+$ and decay order $M > 0$, if there holds

$$\|\mathbf{1}_F K(t,s) \mathbf{1}_E f\|_{L^{r_2}} \lesssim |t-s|^{-1-\frac{n}{m}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \langle \operatorname{dist}(E,F)/|t-s|\rangle_m^{-M} \|\mathbf{1}_E f\|_{L^{r_1}}$$

for all Borel sets $E, F \subset \mathbb{R}^n$, all $(t,s) \in \triangle^c$ and all $f \in L^{r_1} \cap L^2$.

In certain cases, the decay is actually exponential, so the polynomial decay defined as above holds for all $M > 0$, in which case we say that the decay order is ∞.

Definition 2.1.5. Let $m \in \mathbb{N}_+$. Let $1 \le q \le 2$ and $M \in \mathbb{R}_+ \cup \{\infty\}$. We say $T \in SIO^+_{m,q,M}$ if $T \in SIO^+$ and the associated operator-valued kernel $K(t,s)$ satisfies $L^q - L^2$ **and**[1] $L^2 - L^2$ off-diagonal decay with homogeneity m and decay order M.

We do not treat the fractional homogeneity here, namely, the case when $m \in \mathbb{R}_+$. The most important parameters in $SIO^+_{m,q,M}$ are q and M.

Let q' be the dual exponent of $q \in [1,\infty]$, namely, $1/q + 1/q' = 1$. Here, $1/\infty = 0$.

We prove some endpoint weak-type estimates extending results of [AKMP12].

Theorem 2.1.6. *Let $m \in \mathbb{N}_+$ and $\beta < 1$. Let $T \in SIO^+_{m,q,M}$ with $1 \le q \le 2$, $M > \frac{n}{2m}$ and let $p_M < 1$ be given by $M = \frac{n}{m}\left(\frac{1}{p_M} - \frac{1}{2}\right)$. Let q' be the dual exponent of q.*

(1) If $q' \le \frac{2n}{m(1-\beta)}$ or equivalently $\frac{n}{2m} \ge -\frac{\beta-1}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right)$, then T extends to a bounded operator on $T^{p,2,m}_\beta$ when $\infty \ge p > p_c$, where

$$p_c = \frac{4n}{2n + m(1-\beta)q'} \ge 1.$$

(2) If $q' > \frac{2n}{m(1-\beta)}$ or equivalently $\frac{n}{2m} < -\frac{\beta-1}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right)$, then T extends to a bounded operator mapping $T^{\widehat{p}_c,2,m}_\beta$ into ${}^{w}T^{\widehat{p}_c,2,m}_\beta$, thus bounded on $T^{p,2,m}_\beta$ when $\infty \ge p > \widehat{p}_c$, where $\widehat{p}_c = \max\left(p_M, \widetilde{p}_c\right)$ and

$$\widetilde{p}_c = \frac{2n}{\frac{2n}{q} + m(1-\beta)} < 1.$$

In the above theorem, the $T^{p,2,m}_\beta$-boundedness of T for $2 < p \le \infty$ is given in Proposistion 4.2 of [AKMP12]. The remaining intermediate results, namely $T^{p,2,m}_\beta$-boundedness of T for $p_c < p < 2$ in (1) and $\widehat{p}_c < p < 2$ in (2), are obtained in Theorem 3.1 of [AKMP12] by using atomic decompositions of tent spaces. So (1) of the above theorem is the same as [AKMP12], and our contribution is the $T^{\widehat{p}_c,2,m}_\beta$ to ${}^{w}T^{\widehat{p}_c,2,m}_\beta$-boundedness of T in (2). It would be helpful to mention that, the lack of the endpoint claim for $p = p_c$ in (1), is due to the fact that Stein's interpolation for analytic families of operators on quasi-Banach and Lorentz type tent spaces is unknown to us.

As noted in [AKMP12, Section 5], these extrapolation results apply to maximal regularity operators on tent spaces. We shall treat the maximal regularity operator separately in Part II of this thesis.

1. This differs with the definition in [AKMP12] when $q < 2$. We always assume the $L^2 - L^2$ off-diagonal decay of the kernel, which is responsible for the $T^{p,2,m}_\beta$-boundedness of T for $p > 2$.

The organization of this chapter is as follows. In next section, we shall first propose a quite general extrapolation machinery à la Calderón-Zygmund (see Theorem 2.2.1) for sublinear operators acting on tent spaces, by resorting to some suitable Calderón-Zygmund type decomposition lemmata for tent space functions (proved in Chapter 1). We also prove a variant of this extrapolation (see Theorem 2.2.2) by establishing certain Hardy-Littlewood type embeddings for tent space functions. In the third section, we relate our extrapolation method with the one by atomic decompositions of tent spaces, which was used in [AKMP12]. In the next-to-last section, we prove the main result (see Theorem 2.1.6) by using this extrapolation machinery and its variant.

2.2 A tent-space Calderón-Zygmund theory

In this section we propose tent-space variants of the Lebesgue-space Calderón-Zygmund theory pioneered by S. Blunck and P. C. Kunstmann in [BK03] (see X. T. Duong and A. McIntosh [DM99] for earlier references). The Blunck-Kunstmann type Calderón-Zygmund theory concerns the singular but generally non-integral operators on $L^p(\mathbb{R}^n)$ (and enjoying the $L^p(\mathbb{R}^n)$-boundedness only for p in a sub-interval of $(1,\infty)$). We shall adapt the arguments of the version of Blunck-Kunstmann theorem that is presented in the memoir [Aus07].

For a ball $B \subset \mathbb{R}^n$, we let λB be the ball with same center and radius λ times that of B. That is, $r_{\lambda B} = \lambda r_B$. We set the Carleson (tent) annuli

$$C_j(B) = \widehat{2^{j+1}B} \backslash \widehat{2^j B} \text{ if } j \geq 2, \text{ and } C_1(B) = \widehat{4B}.$$

Here the tent above B is defined by $\widehat{B} := \left\{(t,y) \in \mathbb{R}^{1+n}_+ : B(y,t) \subset B\right\}$.

Let $0 < p < \infty$. We say that a sublinear operator T acting on a subspace D of $T^{p,2}$ is of weak-type (p,p) if it maps D boundedly into ${}^wT^{p,2}$, namely

$$\|T(f)\|_{{}^wT^{p,2}} \lesssim \|f\|_{T^{p,2}}, \quad \forall f \in D,$$

and is of strong-type (p,p) if it maps D boundedly into $T^{p,2}$, namely,

$$\|T(f)\|_{T^{p,2}} \lesssim \|f\|_{T^{p,2}}, \quad \forall f \in D.$$

Here, T is "sublinear" means the inequality

$$|T(f+g)| \leq |T(f)| + |T(g)|$$

holds almost everywhere for every f and g in D. Usually, we let $D = D_p = T^{2,2} \cap T^{p,2}$, which is a dense subspace of $T^{p,2}$ for $0 < p < \infty$.

Our first abstract extrapolation result reads as follow.

Theorem 2.2.1. *Assume $M, \widetilde{M} > \frac{n}{2}$. Let $p_M < 1$ be given by $M = n\left(\frac{1}{p_M} - \frac{1}{2}\right)$. Suppose T is a sublinear operator acting on $T^{2,2}$, and T is of weak-type $(2,2)$. Suppose $\{A_r\}_{r>0}$ is a family of linear operators acting on $T^{2,2}$.*

Assume further that for any ball $B \subset \mathbb{R}^n$, for any $j \geq 2$

$$\left(\iint\limits_{C_j(B)} |T(I - A_{r_B})(f)(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2} \leq c2^{-j\widetilde{M}} |B|^{-\frac{M}{n}} \|f\|_{T^{p_M,2}} \quad , \tag{2.2.1}$$

and that for any $j \geq 1$

$$\left(\iint\limits_{C_j(B)} |A_{r_B}(f)(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2} \leq c2^{-j\widetilde{M}} |B|^{-\frac{M}{n}} \|f\|_{T^{p_M,2}} \quad , \tag{2.2.2}$$

for any $f \in T^{2,2}$ supported in the tent $\widehat{B}$.

Then T is of weak-type (p_M, p_M), hence strong-type (p,p) for $p_M < p < 2$, and the operator norm of T depends on n, p_M, c, M and $\|T\|_{T^{2,2} \to {}^wT^{2,2}}$.

It is interesting to note the following technical[2] conditions on T:

For any ball $B \subset \mathbb{R}^n$, assume that for any $j \geq 2$

$$\left(\frac{1}{|2^{j+1}B|} \iint\limits_{C_j(B)} |T(I - A_{r_B})(f)(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2} \leq c2^{-j(\widetilde{M} + \frac{n}{2})} \left(\frac{1}{|B|} \iint\limits_{\widehat{B}} |f(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2},$$

and for any $j \geq 1$

$$\left(\frac{1}{|2^{j+1}B|} \iint\limits_{C_j(B)} |A_{r_B}(f)(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2} \leq c2^{-j(\widetilde{M} + \frac{n}{2})} \left(\frac{1}{|B|} \iint\limits_{\widehat{B}} |f(t,y)|^2 \frac{dt\,dy}{t} \right)^{1/2},$$

and for any $f \in T^{2,2}$ supported in the tent $\widehat{B}$.

These conditions are weaker than (2.2.1) and (2.2.2) (this can be seen via estimating $|B|^{-\frac{M}{n}} \|f\|_{T^{p_M,2}}$ by $\|f\|_{T^{2,2}}$, upon using the Hölder's inequality). Under these conditions, the strong-type (p,p) for $p_{\widetilde{M}} < p < 2$ of a strong type $(2,2)$ operator T comes as an immediate consequence of the extrapolation method by atomic decompositions of tent spaces. We will come back to this issue later in Section 2.3.

2. The technicality lies in the *largeness* ($\widetilde{M} > \frac{n}{2}$) of the order of off-diagonal decay.

The set of off-diagonal decay conditions (2.2.1) and (2.2.2) is not friendly to check due to the term $\|f\|_{T^{p_M,2}}$ in the right hand sides. In applications (2.2.1) and (2.2.2) shall be replaced by some stronger versions (hence the corresponding theorem is weaker), following some Hardy-Littlewood embeddings for tent space functions proved in the third subsection below.

Our second abstract extrapolation result reads as follow.

Theorem 2.2.2. *Assume $M, \widetilde{M} > \frac{n}{2}$. Let $p_M < 1$ be given by $M = n\left(\frac{1}{p_M} - \frac{1}{2}\right)$. For $1 \le q_i \le 2$, $i = 1,2$, let $M_{q_i} = n\left(\frac{1}{p_M} - \frac{1}{q_i}\right)$. Suppose T is a sublinear operator acting on $T^{2,2}$, and T is of weak-type $(2,2)$. Suppose $\{A_r\}_{r>0}$ is a family of linear operators acting on $T^{2,2}$.*

Assume further that for any ball $B \subset \mathbb{R}^n$, for any $j \ge 2$

$$\begin{aligned}\left(\iint_{C_j(B)} |T(I - A_{r_B})(f)(t,y)|^2 \frac{dt dy}{t}\right)^{1/2} \\ \le c2^{-j\widetilde{M}}|B|^{-\frac{M}{n}} \left(\int_0^{r_B}\left(\int_B |f(t,x)|^{q_1}\,dx\right)^{2/q_1} t^{2M_{q_1}}\frac{dt}{t}\right)^{1/2},\end{aligned} \tag{2.2.3}$$

and for any $j \ge 1$

$$\begin{aligned}\left(\iint_{C_j(B)} |A_{r_B}(f)(t,y)|^2 \frac{dt dy}{t}\right)^{1/2} \\ \le c2^{-j\widetilde{M}}|B|^{-\frac{M}{n}} \left(\int_0^{r_B}\left(\int_B |f(t,x)|^{q_2}\,dx\right)^{2/q_2} t^{2M_{q_2}}\frac{dt}{t}\right)^{1/2},\end{aligned} \tag{2.2.4}$$

for any $f \in T^{2,2}$ supported in the tent $\widehat{B}$.

Then T is of weak-type (p_M, p_M), hence strong-type (p,p) for $p_M < p < 2$, and the operator norm of T depends on n, p_M, c, M and $\|T\|_{T^{2,2}\to {}^wT^{2,2}}$.

Remark 2.2.3. Theorems 2.2.1 – 2.2.2 apply to the scales of tent spaces $T^{p,2,m}_\beta$ and ${}^wT^{p,2,m}_\beta$ by considering $\iota^{-1} \circ T \circ \iota$, where ι is the isometry given in Remark 2.1.2.

In the following subsections we prove Theorems 2.2.1 – 2.2.2 for $m = 1$ and $\beta = -1$.

2.2.1 Calderón-Zygmund decompositions (CZD) in tent spaces

We state for convenience Theorem 1.1.1 in Chapter 1.

Lemma 2.2.4. *For any $f \in T^{p,2}$, $0 < p < \infty$, and any height $\lambda > 0$, there exists $C = C(n,p) > 0$ such that we can always find a family of balls $\{B_i\}_{i\in\mathbb{N}_+}$ in $\mathbb{R}^n$ and a Calderón-Zygmund decomposition $f = g + \sum_{i\in\mathbb{N}_+} b_i$, with $\operatorname{supp} b_i \subset \widehat{B_i}$, such that*

$$\|g\|_{T^{\infty,2}} \leq C\lambda, \tag{2.2.5}$$

$$\|b_i\|^p_{T^{p,2}} \leq C\lambda^p |B_i|, \tag{2.2.6}$$

and

$$\sum_{i\in\mathbb{N}_+} |B_i| \leq C\lambda^{-p}\|f\|^p_{T^{p,2}}. \tag{2.2.7}$$

Moreover, there exists $1 > \beta = \beta(n) > 0$ such that $\{\beta B_i\}$ are disjoint, namely,

$$\sum_{i\in\mathbb{N}_+} \mathbf{1}_{\beta B_i} \leq 1. \tag{2.2.8}$$

These Calderón-Zygmund decompositions are proved in Chapter 1.

2.2.2 Proof of Theorem 2.2.1 via (CZD)

By density of $T^{p_M,2} \cap T^{2,2}$ in $T^{p_M,2}$, it suffices to prove that, for any $f \in T^{p_M,2} \cap T^{2,2}$ and any $\lambda > 0$ there hold the following weak-type estimates

$$\left|\left\{x \in \mathbb{R}^n \middle| \mathscr{A}(T(f))(x) > \lambda\right\}\right| \lesssim \frac{1}{\lambda^{p_M}} \int_{\mathbb{R}^n} \mathscr{A}(f)^{p_M}(x)\,dx.$$

Recall that the functional $\mathscr{A}$ is defined in (2.1.1) (for $m = 1$ and $\beta = -1$). Apply the Calderón-Zygmund decomposition $f = g + \sum_{i\in\mathbb{N}_+} b_i$ at height λ, with $\operatorname{supp} b_i \subset \widehat{B}_i$, r_i the radius of B_i. Let $B_r = I - A_r$. By sublinearity of the operator T and the functional $\mathscr{A}$, we have for every $x \in \mathbb{R}^n$

$$\begin{aligned}\mathscr{A}(T(f))(x) &\leq \mathscr{A}(T(g))(x) + \mathscr{A}\left(T\left[\sum_{i\in\mathbb{N}_+} A_{r_i}(b_i)\right]\right)(x) + \mathscr{A}\left(T\left[\sum_{i\in\mathbb{N}_+} B_{r_i}(b_i)\right]\right)(x)\\ &=: G_1(x) + G_2(x) + G_3(x),\end{aligned}$$

then the proof reduces to check that $G_k \in L^{p_M,\infty}$, $k = 1,2,3$.

By the weak-type $(2,2)$ of T and the interpolation control for g from the Calderón-Zygmund decomposition, we have for G_1

$$\begin{aligned}\left|\left\{x \in \mathbb{R}^n \middle| G_1(x) > \lambda/3\right\}\right| &\lesssim \lambda^{-2}\|T(g)\|^2_{wT^{2,2}}\\ &\lesssim \lambda^{-2}\|g\|^2_{T^{2,2}} \lesssim \lambda^{-p_M}\|f\|^{p_M}_{T^{p_M,2}}.\end{aligned}$$

This proves $G_1 \in L^{p_M,\infty}$.

By the weak-type $(2,2)$ of T, we have for G_2 that

$$\left|\left\{x \in \mathbb{R}^n \middle| G_2(x) > \lambda/3\right\}\right| \leq \lambda^{-2}\left\|T\left(\sum_{i\in\mathbb{N}_+} A_{r_i}(b_i)\right)\right\|^2_{wT^{2,2}} \lesssim \lambda^{-2}\left\|\sum_{i\in\mathbb{N}_+} A_{r_i}(b_i)\right\|^2_{T^{2,2}}.$$

Let $C_{ij} = C_j(B_i)$. For $j \geq 1$, by decay assumptions (2.2.2)

$$\begin{aligned}
\left\| A_{r_i}(b_i) \right\|_{T^{2,2}(C_{ij})} &= \left(\iint_{C_{ij}} |A_{r_i}(b_i)(t,y)|^2 \frac{dt dy}{t} \right)^{1/2} \\
&\lesssim 2^{-j\widetilde{M}} |B_i|^{-\frac{M}{n}} \left(\int_{B_i} |\mathscr{A}(b_i)(x)|^{p_M} dx \right)^{\frac{1}{p_M}} \\
&\lesssim |B_i|^{1/2} 2^{-j\widetilde{M}} \lambda.
\end{aligned}$$

In the final step, we use the $T^{p_M,2}$ quasi-norm estimates on the bad functions b_i from the Calderón-Zygmund decomposition.

Now we dualize with $u \in T^{2,2}$, and $\|u\|_{T^{2,2}} = 1$. Then for any $j \geq 1$ and any $\widetilde{y} \in \beta B_i$, with $\beta < 1$ given in Lemma 2.2.4, we have

$$\begin{aligned}
&\left| \iint_{C_{ij}} u(t,y) \left| A_{r_i}(b_i)(t,y) \right| \frac{dt dy}{t} \right| \\
&\leq \left\| A_{r_i}(b_i) \right\|_{T^{2,2}(C_{ij})} \|u\|_{T^{2,2}(C_{ij})} \\
&\lesssim |B_i|^{1/2} 2^{-j\widetilde{M}} \lambda \left\| \mathscr{A}(u)|_{2^{j+1}B_i} \right\|_{L^2} \\
&\lesssim |B_i|^{1/2} 2^{-j\widetilde{M}} \lambda \left| 2^{j+1} B_i \right|^{1/2} \mathbf{M} \left[\mathscr{A}(u)^2 \right]^{1/2} (\widetilde{y}) \\
&\lesssim \lambda 2^{(\frac{n}{2} - \widetilde{M})j} |B_i| \mathbf{M} \left[\mathscr{A}(u)^2 \right]^{1/2} (\widetilde{y}).
\end{aligned}$$

Here $\mathbf{M}$ is the non-centered maximal function in $\mathbb{R}^n$. Taking an average on βB_i, then

$$\iint_{C_{ij}} u(t,y) A_{r_i}(b_i)(t,y) \frac{dt dy}{t} \lesssim \lambda 2^{(\frac{n}{2} - \widetilde{M})j} \int_{\beta B_i} \mathbf{M} \left[\mathscr{A}(u)^2 \right]^{1/2} (y) dy.$$

Summing over $j \geq 1$ and $i \in \mathbb{N}_+$, and using $\sum_{i \in \mathbb{N}_+} \mathbf{1}_{\beta B_i} \leq 1$, we have

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{1+n}_+} u(t,y) \sum_{i \in \mathbb{N}_+} \left| A_{r_i}(b_i)(t,y) \right| \frac{dt dy}{t} \right| \\
&= \left| \sum_{i \in \mathbb{N}_+} \sum_{j \geq 1} \iint_{C_{ij}} u(t,y) \left| A_{r_i}(b_i)(t,y) \right| \frac{dt dy}{t} \right| \\
&\lesssim \lambda \int_{\cup_i \beta B_i} \sum_{i \in \mathbb{N}_+} \mathbf{1}_{\beta B_i}(y) \mathbf{M} \left[\mathscr{A}(u)^2 \right]^{1/2} (y) dy \\
&\lesssim \lambda \int_{\cup_i \beta B_i} \mathbf{M} \left[\mathscr{A}(u)^2 \right]^{1/2} (y) dy
\end{aligned}$$

$$\lesssim \lambda|\cup_i B_i|^{1/2} \left\|\mathscr{A}(u)^2\right\|_{L^1}^{1/2}.$$

In the next to last inequality, we use the so-called Kolmogorov's lemma (see [Duo01, Lemma 5.16] for a statement and for its application see [HM03] and [Aus04]) and the fact that $\mathbf{M}$ is bounded from L^1 into $L^{1,\infty}$. Thus

$$\left\|\sum_{i\in\mathbb{N}_+} \left|A_{r_i}(b_i)\right|\right\|_{T^{2,2}} \lesssim \lambda|\cup_i B_i|^{1/2},$$

and then we obtain the weak-type estimate

$$\left|\left\{x\in\mathbb{R}^n \middle| G_2 > \lambda/3\right\}\right| \lesssim \lambda^{-p_M}\|f\|_{T^{p_M,2}}^{p_M}.$$

This proves $G_2 \in L^{p_M,\infty}$.

By the properties of the Calderón-Zygmund decomposition, we have for G_3

$$\begin{aligned}
&\left|\left\{x\in\mathbb{R}^n \middle| G_3(x) > \lambda/3\right\}\right| \\
&\quad \le |\cup_i 4B_i| + \left|\left\{x\in \complement(\cup_i 4B_i) \middle| G_3(x) > \lambda/3\right\}\right| \\
&\quad \lesssim \lambda^{-p_M}\|f\|_{T^{p_M,2}}^{p_M} + \lambda^{-2}\left\|\mathbf{1}_{\complement(\widehat{\cup_i 4B_i})} \sum_{i\in\mathbb{N}_+}\left|TB_{r_i}(b_i)\right|\right\|_{{}^wT^{2,2}}^2 \\
&\quad \lesssim \lambda^{-p_M}\|f\|_{T^{p_M,2}}^{p_M} + \lambda^{-2}\left\|\sum_{i\in\mathbb{N}_+}\mathbf{1}_{\complement(\widehat{4B_i})}\left|TB_{r_i}(b_i)\right|\right\|_{{}^wT^{2,2}}^2 \\
&\quad \lesssim \lambda^{-p_M}\|f\|_{T^{p_M,2}}^{p_M} + \lambda^{-2}\left\|\sum_{i\in\mathbb{N}_+}\mathbf{1}_{\complement(\widehat{4B_i})}\left|TB_{r_i}(b_i)\right|\right\|_{T^{2,2}}^2,
\end{aligned}$$

where $\complement(\cdot)$ is the complement in $\mathbb{R}^n$ or $\mathbb{R}^{1+n}_+$. Note that there needs more details in obtaining the second estimate, namely,

$$\left|\left\{x\in \complement(\cup_i 4B_i) \middle| G_3(x) > \lambda/3\right\}\right| \lesssim \lambda^{-2}\left\|\mathbf{1}_{\complement(\widehat{\cup_i 4B_i})} \sum_{i\in\mathbb{N}_+}\left|TB_{r_i}(b_i)\right|\right\|_{T^{2,2}}^2. \tag{Conv}$$

First, the above arguments on G_2 show the convergence of $\sum_{i\in\mathbb{N}_+} A_{r_i}(b_i)$ in $T^{2,2}$. By inspection of the arguments in establishing the Calderón-Zygmund decompositions of Chapter 1, the series $\sum_{i\in\mathbb{N}_+} b_i$ also converges in $T^{2,2}$ as $f\in T^{2,2}$. Thus by difference, this shows the convergence of $\sum_{i\in\mathbb{N}_+} B_{r_i}(b_i)$ in $T^{2,2}$. Then, observe that for any F

$$\mathbf{1}_{\complement(\cup_i 4B_i)}\mathscr{A}(F) \le \mathscr{A}\left(\mathbf{1}_{\complement(\widehat{\cup_i 4B_i})}F\right).$$

Thus

$$\begin{aligned}
&\left|\left\{x\in \complement(\cup_i 4B_i) \middle| G_3(x) > \lambda/3\right\}\right| \\
&\quad \le \left|\left\{x\in\mathbb{R}^n \middle| \mathscr{A}\left(\mathbf{1}_{\complement(\widehat{\cup_i 4B_i})} T\left[\sum_{i\in\mathbb{N}_+} B_{r_i}(b_i)\right]\right)(x) > \lambda/3\right\}\right|
\end{aligned}$$

$$\lesssim \lambda^{-2} \left\| \mathbf{1}_{\complement(\widehat{\cup_i 4B_i})} T\left[\sum_{i\in\mathbb{N}_+} B_{r_i}(b_i) \right] \right\|^2_{{}^wT^{2,2}}$$
$$\lesssim \lambda^{-2} \left\| \mathbf{1}_{\complement(\widehat{\cup_i 4B_i})} \left[\sum_{i}^{N} \left| TB_{r_i}(b_i) \right| \right] \right\|^2_{{}^wT^{2,2}} + \lambda^{-2} \left\| T(\beta_N) \right\|^2_{{}^wT^{2,2}},$$

where $\beta_N = \sum_{i>N} B_{r_i}(b_i)$ and we profited from the sublinearity of T to use

$$\left| T\left[\sum_{i\in\mathbb{N}_+} B_{r_i}(b_i) \right] \right| \leq \sum_{i=1}^{N} \left| TB_{r_i}(b_i) \right| + |T(\beta_N)|$$

in the final step. Since $\beta_N \to 0$ in $T^{2,2}$, by the weak-type $(2,2)$ of T we have $\left\| T(\beta_N) \right\|_{{}^wT^{2,2}} \to 0$, and the estimate (Conv) is proved.

Now we can carry out the above arguments for G_2 similarly with respect to G_3, namely, to estimate $\left\| \sum_{i\in\mathbb{N}_+} \mathbf{1}_{\complement(\widehat{4B_i})} \left| TB_{r_i}(b_i) \right| \right\|^2_{T^{2,2}}$, with the same duality technique, using the set of decay assumptions (2.2.1), and with a summation over $j \geq 2$. This observation leads to $G_2 \in L^{p_M,\infty}$.

In all, this proves the weak-type (p_M, p_M) of T. The proof of strong-type (p,p) for $p \in (p_M, 2)$ follows from Marcinkiewicz interpolation theorem (applied to $\mathscr{A} \circ T$).

2.2.3 Hardy-Littlewood embeddings (HLE) for tent space functions

Let $0 < p < 2$. Let $\beta_2(p) = 2n\left(\frac{1}{p} - \frac{1}{2}\right) - 1$. Define the weighted space $L^2_{\beta_2(p)}\left(L^2_x\right)$ as the set of L^2_{loc} functions such that

$$\|f\|_{L^2_{\beta_2(p)}(L^2_x)} := \left(\int_0^\infty \left(\int_{\mathbb{R}^n} |f(t,x)|^2 \, dx \right) t^{2n\left(\frac{1}{p}-\frac{1}{2}\right)} \frac{dt}{t} \right)^{1/2} < \infty.$$

This is a Banach space though p could be less than 1.

For $x \in \mathbb{R}^n$, let $\Gamma(x) = \left\{ (t,y) \in \mathbb{R}^{1+n}_+ ; B(y,t) \ni x \right\}$. For $f \in C\left(\mathbb{R}^{1+n}_+\right)$, the class of continuous functions on $\mathbb{R}^{1+n}_+$, let

$$\mathscr{N}_*(f)(x) = \sup_{(t,y)\in\Gamma(x)} |f(t,y)|, \quad x \in \mathbb{R}^n.$$

Let $W(t,y) = (t/2, 2t) \times B(y,t)$ be the Whitney box centered at $(t,y) \in \mathbb{R}^{1+n}_+$. Let $1 \leq r < \infty$. Define the non-tangential maximal functional $\widetilde{\mathscr{N}}_*^r$ by

$$\widetilde{\mathscr{N}}_*^r(f)(x) = \sup_{(t,y)\in\Gamma(x)} \left(\frac{1}{|W(t,y)|} \iint_{W(t,y)} |f(s,z)|^r \, ds dz \right)^{1/r}, \quad x \in \mathbb{R}^n.$$

Let $\widetilde{\mathscr{N}}_* = \widetilde{\mathscr{N}}_*^2$.

Similar to Definition 2.1.1, we can define scales of tent spaces by using the non-tangential maximal functional $\widetilde{\mathcal{N}}_*^r$ instead of the area functional $\mathscr{A}$. We have the following versions of Hardy-Littlewood embeddings for tent space functions (see [CT75] and [Tor86] for further informations on these embedding inequalities in connection with the classical Hardy space theory).

Lemma 2.2.5. *If* $0 < p < 2$ *and* $\widetilde{\mathcal{N}}_*(f) \in L^p$, *then*

$$\|f\|_{L^2_{\beta_2(p)}(L^2_x)} \lesssim \left\|\widetilde{\mathcal{N}}_*(f)\right\|_{L^p}. \tag{2.2.9}$$

This elementary embedding result was first obtained in [HMM13] in the setting where $\mathbb{R}^{1+n}_+$ is replaced by a Lipschitz graph domain. Their main arguments aim at factorizing a Carleson measure from f. Here we give another approach by using atomic decompositions of tent spaces.

Proof. Taking the square, $g = |f|^2$, it suffices to show

$$\|g\|_{L^1_{\beta_2(p)}(L^1_x)} \lesssim \left\|\widetilde{\mathcal{N}}_*^1(g)\right\|_{L^{p/2}}. \tag{2.2.10}$$

Here $L^1_{\beta_2(p)}\left(L^1_x\right) := L^1\left(\mathbb{R}_+, t^{\beta_2(p)}dt; L^1(\mathbb{R}^n)\right)$.

Taking the Whitney average, $h = \frac{1}{|W(t,y)|}\iint_{W(t,y)} g$, then it suffices to show

$$\|h\|_{L^1_{\beta_2(p)}(L^1_x)} \lesssim \|\mathcal{N}_*(h)\|_{L^{p/2}}. \tag{2.2.11}$$

Note that as an average, h has the needed continuity in $\mathbb{R}^{1+n}_+$.

Now since $p/2$ is in the atomic range (by which we mean $p/2 < 1$), we can use the atomic decomposition[3] for h with $\|\mathcal{N}_*(h)\|_{L^{p/2}} < \infty$, so that $h = \sum_j \lambda_j a_j$ where a_j are atoms, namely, supported in some tent $\widehat{B_j}$ with $\sup|a_j| \le |B_j|^{-2/p}$, and there holds

$$\left\|\{\lambda_j\}\right\|_{l^{p/2}} \lesssim \|\mathcal{N}_*(h)\|_{L^{p/2}}.$$

Taking such an atom a, supported in $(0, r_B) \times B$, with $\sup|a| \le |B|^{-2/p}$, it is straightforward to see that

$$\|a\|_{L^1_{\beta_2(p)}(L^1_x)} \lesssim \left(|B|^{\left(-\frac{2}{p}\right)+1}\right) r_B^{2n\left(\frac{1}{p}-\frac{1}{2}\right)} \lesssim 1.$$

The conclusion of this lemma follows since the above estimates on atoms are uniform and we have the classical inequality $l^{p/2} \hookrightarrow l^1$. □

Now we consider a variant of Lemma 2.2.5.

Let $0 < q \le 2$. For $0 < p < q$ define the space $L^2_{\beta_q(p)}\left(L^q_x\right)$ as the set of $L^q_{\mathrm{loc}}\left(\mathbb{R}^{1+n}_+\right)$ functions such that

$$\|f\|_{L^2_{\beta_q(p)}(L^q_x)} := \left(\int_0^\infty \left(\int_{\mathbb{R}^n} |f(t,x)|^q\, dx\right)^{2/q} t^{2n\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{dt}{t}\right)^{1/2} < \infty,$$

with weight $\beta_q(p) = 2n\left(\frac{1}{p} - \frac{1}{q}\right) - 1$.

3. This is an adaption of [CMS85, Proposition 2].

Lemma 2.2.6. *If $0 < p < q \le 2$ and $\widetilde{\mathcal{N}_*}(f) \in L^p$, then*

$$\|f\|_{L^2_{\beta_q(p)}(L^q_x)} \lesssim \left\|\widetilde{\mathcal{N}_*}(f)\right\|_{L^p}. \tag{2.2.12}$$

Proof. Let g be the square of f, it suffices to show

$$\|g\|_{L^1_{\beta_q(p)}\left(L^{q/2}_x\right)} \lesssim \left\|\widetilde{\mathcal{N}^1_*}(g)\right\|_{L^{p/2}}. \tag{2.2.13}$$

Here $L^1_{\beta_q(p)}\left(L^{q/2}_x\right) := L^1\left(\mathbb{R}_+, t^{\beta_q(p)}dt; L^{q/2}(\mathbb{R}^n)\right)$.

For each $t > 0$, and $g_t = g(t,\cdot)$, using $q \le 2$ and Hölder's inequality,

$$\left(\int_{\mathbb{R}^n} |g_t|^{q/2}\right)^{2/q} = \left(\int_{\mathbb{R}^n}\fint_{B(x,t)} |g_t|^{q/2}dx\right)^{2/q} \le \left(\int_{\mathbb{R}^n}\left(\fint_{B(x,t)}|g_t|\right)^{q/2}dx\right)^{2/q}.$$

Inserting this into the vertical integral $\int_0^\infty(\cdot)\,t^{\beta_q(p)}dt$ and using Minkowski inequality,

$$\|g\|_{L^1_{\beta_q(p)}\left(L^{q/2}_x\right)} \le \left(\int_{\mathbb{R}^n}\left(\int_0^\infty\left(\fint_{B(x,t)}|g_t|\right)t^{\beta_q(p)}dt\right)^{q/2}dx\right)^{2/q},$$

then for (2.2.13) it suffices to show

$$\left\|\left(\iint\!\!\!\!\!\!-_{W(t,x)} g\right)\right\|_{L^{q/2}_x\left(L^1_{\beta_q(p)}\right)} \lesssim \left\|\mathcal{N}_*\left(\iint\!\!\!\!\!\!-_{W(t,x)} g\right)\right\|_{L^{p/2}}, \tag{2.2.14}$$

Here $L^{q/2}_x\left(L^1_{\beta_q(p)}\right) := L^{q/2}\left(\mathbb{R}^n; L^1(\mathbb{R}_+, t^{\beta_q(p)}dt)\right)$.

Taking the Whitney averages, $h = \frac{1}{|W(t,x)|}\iint_{W(t,x)} g$, (2.2.14) reduces to show

$$\left\|t^{\beta_q(p)}h\right\|_{L^{q/2}_x\left(L^1_{\beta_q(p)}\right)} \lesssim \|\mathcal{N}_*(h)\|_{L^{p/2}}, \tag{2.2.15}$$

Note that as an average, h has the needed continuity in $\mathbb{R}^{1+n}_+$.

Now since $p/2$ is in the atomic range, we can use the atomic decomposition for h as before. Taking such an atom a, supported in $(0, r_B)\times B$, with $\sup|a| \le |B|^{-2/p}$, it is straightforward to see that

$$\|a\|_{L^{q/2}_x\left(L^1_{\beta_q(p)}\right)} \lesssim |B|^{-2/p}|B|^{2/q} r_B^{2n\left(\frac{1}{p}-\frac{1}{q}\right)} \lesssim 1.$$

The conclusion follows since the above estimates on atoms of h are uniform. □

Corollary 2.2.7. *If $0 < p < q \le 2$ and $\mathcal{A}(f) \in L^p$, then*

$$\|f\|_{L^2_{\beta_q(p)}(L^q_x)} \lesssim \left\|\mathcal{A}(f)\right\|_{L^p}. \tag{2.2.16}$$

Proof. Indeed, first we has the elementary observation

$$\widetilde{\mathcal{N}_*}(f)(x) \lesssim \mathcal{A}^{(\alpha)}(f)(x), \quad x \in \mathbb{R}^n,$$

for some $\alpha > 1$, where

$$\mathcal{A}^{(\alpha)}(f)(x) := \left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,\alpha t)}(y)}{t^n} |f(t,y)|^2 \frac{dt dy}{t} \right)^{1/2}.$$

Next we have

$$\left\| \mathcal{A}^{(\alpha)}(f) \right\|_{L^p} \lesssim \left\| \mathcal{A}(f) \right\|_{L^p}, \quad 0 < p < \infty, \tag{2.2.17}$$

which is the quasi-norm equivalence under change of apertures in tent spaces. See for example [Aus11], or [Tor86, Uch01].

We conclude using Lemma 2.2.6. □

2.2.4 Proof of Theorem 2.2.2 via Theorem 2.2.1 and (HLE)

It suffices to compare the right hand sides of the off-diagonal decay conditions in Theorem 2.2.2 and Theorem 2.2.1, and use the above Corollary 2.2.7.

2.3 Relation with the extrapolation method by atomic decompositions of tent spaces

We relate our Calderón-Zygmund extrapolation methods to the extrapolation method by atomic decompositions which was employed in [AMR08, AKMP12].

Theorem 2.3.1. *Let $M > \frac{n}{2}$ and $p_M < 1$ be given by $M + \frac{n}{2} = \frac{n}{p_M}$. Let $p_0 \in (0,2]$ and $\widetilde{p} = \min(p_0, p_M)$. Suppose that T is a sublinear operator acting on $T^{2,2}$ and is of strong-type $(2,2)$. Assume for any ball $B \subset \mathbb{R}^n$ and any $j \geq 2$ that*

$$\begin{aligned} &\left(\frac{1}{|2^{j+1}B|} \iint_{C_j(B)} |T(f)(t,y)|^2 \frac{dt dy}{t} \right)^{1/2} \\ &\leq c 2^{-j(M+\frac{n}{2})} \left(\frac{1}{|B|} \int_B |\mathcal{A}(f)(x)|^{p_0} dx \right)^{1/p_0} \end{aligned} \tag{2.3.1}$$

holds for all $f \in T^{2,2}$ supported in $\widehat{B}$.

Then T is of weak-type (p_0, p_0) and is of strong-type (p,p) for $\widetilde{p} < p \leq 2$. The operator norm of T depends on n, p_0, c, M and $\|T\|_{T^{2,2} \to T^{2,2}}$.

Proof. For the weak-type (p_0, p_0) and the strong-type (p, p) for $p \in (p_0, 2)$, we can apply the above Theorem 2.2.1 directly with $A_{r_B} = 0$. Be aware that M and p_0 here correspond to $\widetilde{M}$ and p_M in Theorem 2.2.1.

For the remaining part, namely, strong-type (p, p) for $p_M < p < 2$, we follow the tent space boundedness criteria via atoms (see Lemma 3.3 in [AKMP12], or Step 3 of the proof of Theorem 4.9 in [AMR08]). As $M + \frac{n}{2} > n$ implies $p_M < 1$, we only need to prove for $p \in (p_M, 1]$ that the strong-type $(2,2)$ operator T maps $T^{p,2}$-atoms into $T^{p,2}$ with a uniform operator norm[4]. Then the strong-type (p, p) of T comes by an extension process from the atomic decompositions of $T^{p,2} \cap T^{2,2}$.

Suppose now f is a $T^{p,2}$-atom supported in $\widehat{B}$. Then combining the estimate

$$\left(\frac{1}{|B|}\int_B |\mathscr{A}(f)(x)|^{p_0}\,dx\right)^{1/p_0} \le \left(\frac{1}{|B|}\int_B |\mathscr{A}(f)(x)|^2\,dx\right)^{1/2} \le |B|^{-1/p} \tag{2.3.2}$$

with the decay assumption (2.3.1), letting $\lambda_j \sim 2^{jn(\frac{1}{p}-\frac{1}{p_M})}$, we know that $\lambda_j^{-1} T(f)|_{C_j(B)}$ is a $T^{p,2}$-atom supported in $C_j(B)$. Thus when $1 \ge p > p_M$, the sequence $\{\lambda_j\}_{j=2}^{\infty}$ is in l^p, with the l^p quasi-norm independent on f. In another aspect, by Hölder's inequality and $T^{2,2}$-boundedness of T we have

$$\begin{aligned}
\| T(f)\,|_{C_1(B)} \|_{T^{p,2}} &\le |4B|^{\frac{1}{p}-\frac{1}{2}} \| T(f)\,|_{C_1(B)} \|_{T^{2,2}} \\
&\lesssim |4B|^{\frac{1}{p}-\frac{1}{2}} \|f\|_{T^{2,2}} \\
&\lesssim |4B|^{\frac{1}{p}-\frac{1}{2}} |B|^{\frac{1}{2}-\frac{1}{p}} \lesssim 1,
\end{aligned}$$

which is also independent on f. This shows that T is uniformly bounded on $T^{p,2}$-atoms, and as noted above, also bounded on $T^{p,2}$.

By interpolation with the $T^{2,2}$-boundedness of T, we proved strong-type (p, p) of T for $p \in (p_M, 2)$, hence the proof of the theorem is complete. □

Remark 2.3.2. As seen in the use of Hölder's inequality in (2.3.2), the off-diagonal decay condition (2.3.1) can be weakened to

$$\begin{aligned}
&\left(\frac{1}{|2^{j+1}B|}\iint_{C_j(B)} |T(f)(t,y)|^2 \frac{dt\,dy}{t}\right)^{1/2} \\
&\qquad \le c 2^{-j(M+\frac{n}{2})} \left(\frac{1}{|B|}\int_B |\mathscr{A}(f)(x)|^2\,dx\right)^{1/2}
\end{aligned}$$

if $p_M < p_0$. In other words, the Calderón-Zygmund part in the above theorem (namely, weak-type (p_0, p_0)) is needed only when $p_0 \le p_M$, thereby p_0 is smaller than 1. This

4. With the uniform boundedness on $T^{p,2}$-atoms, one can assume that T is only of weak-type $(2,2)$ to deduce $T^{p,2}$-boundedness. This can be verified via arguments similar to [JY10, Lemma 5.1] and [HMM11, Lemma 3.8], in the Hardy space setting, applied to $|T|$ which is non-negative and sublinear. However, in obtaining the uniform boundedness on atoms, one usually need the strong-type $(2,2)$ of T.

is the main difference between our theory in Theorem 2.2.1 and the Lebesgue-space Calderón-Zygmund theory as in [BK03, Aus07], the latter theory being effective in obtaining weak type Lebesgue-space estimates for some endpoint not less than 1.

2.4 Proof of Theorem 2.1.6 via Theorems 2.2.1-2.2.2

This section is devoted to the proof of Theorem 2.1.6 using Theorem 2.2.1 and (HLE), or using Theorem 2.2.2 directly. First we recall

Lemma 2.4.1. *Let $T \in SIO^+$. Then T is bounded on $T^{2,2,m}_{\beta}$ for $\beta < 1$.*

This weighted result is proved in [AA11], based on Schur type estimates and the assumption $T \in SIO^+$. See also a previous article [HK07]. Note that the homogeneity parameter m in the above lemma is not important, since by Fubini's theorem

$$T^{2,2,m}_{\beta} \simeq L^2\left(\mathbb{R}_+, t^{\beta}dt; L^2(\mathbb{R}^n)\right).$$

Recall that in Theorem 2.1.6, the $T^{p,2,m}_{\beta}$-boundedness of T for $2 < p \leq \infty$ is already given in Proposition 4.2 of [AKMP12], by using the $L^2 - L^2$ off-diagonal decay of the kernel. With Lemma 2.4.1, Theorem 2.1.6 follows immediately from the two lemmas below applied to the decomposition of $T \in SIO^+_{m,q,M}$ into its regular part

$$T_2(f)(t) = \int_0^{t/2} K(t,s)f(s)ds$$

plus its singular part $T_1 = T - T_2$.

Lemma 2.4.2. *T_1 extends to a bounded operator mapping $T^{p_M,2,m}_{\beta}$ into ${}^{w}T^{p_M,2,m}_{\beta}$.*

Lemma 2.4.3. *The statement of Theorem 2.1.6 (for the part $p < 2$) holds for T_2.*

To prove these lemmas, we shall use the version of Theorems 2.2.1-2.2.2 with homogeneity m and weight β, and with $A_{r_B} = 0$, $\widetilde{M} = M$.

Thus we have to check the condition:

For any ball $B \subset \mathbb{R}^n$ and any $j \geq 2$

$$\left(\iint_{C_j(B)} |T(f)(t,y)|^2 t^{\beta} dt dy\right)^{1/2}$$

$$(*) \qquad \lesssim 2^{-jM}|B|^{-\frac{M}{n}}\left(\int_{[0,r_B^m]}\left(\int_B |f(t,y)|^q dy\right)^{2/q} t^{2M_q} t^{\beta} dt dy\right)^{1/2}$$

$$\left(\lesssim 2^{-jM}|B|^{-\frac{M}{n}}\|f\|_{T^{p_M,2,m}_{\beta}}\right),$$

holds for any f supported in the Carleson box $\left(0, r_B^m\right] \times B$, where $B = B(x_B, r_B)$.

Here $1 \le q \le 2$, $M_q = n\left(\frac{1}{p_M} - \frac{1}{q}\right)$ with $p_M < 1$ given by $M = n\left(\frac{1}{p_M} - \frac{1}{2}\right)$, and the Carleson (cylinder) annuli are

$$C_j(B) = \left(\left(0, 2^{(j+1)m} r_B^m\right] \times 2^{j+1} B\right) \Big\backslash \left(\left(0, 2^{jm} r_B^m\right] \times 2^j B\right).$$

Note that for simplicity of the arguments below we changed the tents to Carleson cylinders. We also point our that the latter part of $(*)$ follows from Corollary 2.2.7.

2.4.1 Proof of Lemma 2.4.2

First of all, T_1 is of strong-type $(2,2)$ (here, T_1 is $T_\beta^{2,2,m}$-bounded for $\beta < 1$). Now we check the first inequality in condition $(*)$ with $q = 2$.

Now we assume that f is supported in $\left(0, r_B^m\right] \times B(x_B, r_B)$. Then we remark that if $t > 2r_B^m$, $T_2(f)(t,\cdot) = T(f)(t,\cdot)$ because of the definition of T_2 and the support of f. Hence $T_1(f)(t,\cdot) = 0$ for $t > 2r_B^m$. For $j \ge 2$ we let $f_j(t,y) = T_1(f)(t,y)$ if $2^j r_B \le |y - x_B| < 2^{j+1} r_B$, 0 elsewhere, and $f_1(t,y) = T_1(f)(t,y)$ if $|y - x_B| < 2r_B$, 0 elsewhere.

For $j \ge 2$, we have

$$\begin{aligned}
&\int_0^{2^{(j+1)m} r_B^m} \int_{2^j r_B \le |y - x_B| < 2^{j+1} r_B} |f(t,y)|^2 t^\beta \, dt\, dy \\
&= \int_0^{2r_B^m} \int_{2^j r_B \le |y - x_B| < 2^{j+1} r_B} |T_1(f)(t,y)|^2 \, dy\, t^\beta dt \\
&= \int_0^{2r_B^m} \int_{2^j r_B \le |y - x_B| < 2^{j+1} r_B} \left| \int_{t/2}^{t} \left(K(t,s) f(s,\cdot)\right)(y)\, ds \right|^2 dy\, t^\beta dt \\
&\le \int_0^{2r_B^m} \int_{2^j \le |y - x_B| < 2^{j+1} r_B} \int_{t/2}^{t} \left| \left(K(t,s) f(s,\cdot)\right)(y) \right|^2 ds\, dy\, t^\beta dt \\
&\lesssim \int_0^{2r_B^m} \int_{t/2}^{t} \frac{1}{(t-s)} \left(1 + \frac{2^j r_B}{t-s}\right)^{-2M} \|f(s,\cdot)\|_2^2 \, ds\, t^\beta dt \\
&\lesssim 2^{-2mMj} r_B^{-2M} \int_0^{r_B^m} \|f(s,\cdot)\|_2^2 s^\beta \int_s^{2s} \frac{1}{(t-s)} \left(1 + \frac{1}{t-s}\right)^{-2M} dt\, ds \\
&\lesssim 2^{-2mMj} r_B^{-2M} \int_0^{r_B^m} \|f(s,\cdot)\|_2^2 s^{2M} s^\beta ds
\end{aligned}$$

$$\lesssim 2^{-2mMj}|B|^{-2\frac{M}{n}}\|f\|^2_{T^{p_M,2,m}_\beta}.$$

The last estimate follows from Corollary 2.2.7 (extended to general m and β). We remark that the use of the representation

$$T_1(f)(t,y)=\int_{t/2}^{t}\left(K(t,s)f(s,\cdot)\right)(y)ds$$

on the support of f, which is justified by [AKMP12, Lemma 2.3] and the estimates from the last five lines.

We have proved $(*)$ and may apply either Theorem 2.2.1 or Theorem 2.2.2.

Remark 2.4.4. The arguments here differ with and also simplifies the one in the proof of [AKMP12, Lemma 3.4], as we only use the L^2-L^2 off-diagonal decay of the kernel.

2.4.2 Proof of Lemma 2.4.3

The proof adapts the one in [AKMP12] and we use Theorems 2.2.1-2.2.2 instead of the extrapolation by atomic decompositions of tent spaces at the end. Now we check condition $(*)$ with q given in $T\in SIO^+_{m,q,M}$.

First, by inspection of [AKMP12], we know that the exponential order decay of the kernel does not impact the extrapolation of the regular part of T. For simplicity, we assume that $M>\frac{n}{2m}$ is a finite number.

We embed T_2 into an analytic family of integral operators

$$\mathscr{I}_\alpha(f)(t,y):=\int_0^{t/2}\left(\frac{s}{t}\right)^\alpha\left(K(t,s)f(s,\cdot)\right)(y)ds,\quad \alpha\in\mathbb{C}.$$

Observe that

$$\begin{aligned}&\iint_{\mathbb{R}^{1+n}_+}\left|\mathscr{I}_\alpha(f)(t,y)\right|^2 t^\beta dtdy\\ &=\iint_{\mathbb{R}^{1+n}_+}\left|\int_0^{t/2}\left(\frac{s}{t}\right)^{\alpha-\frac{\beta-1}{2}}\left(tK(t,s)\left(s^{\frac{\beta+1}{2}}f(s,\cdot)\right)\right)(y)\frac{ds}{s}\right|^2\frac{dtdy}{t}.\end{aligned}$$

Using Schur's lemma, $t\simeq t-s$ as $s\in(0,t/2)$, and the uniform boundedness of $tK(t,s)$, shows that, provided $\mathrm{Re}\,\alpha-\frac{\beta-1}{2}>0$, the last integral is bounded by

$$C\left(\mathrm{Re}\,\alpha-\frac{\beta-1}{2}\right)\iint_{\mathbb{R}^{1+n}_+}\left|s^{\frac{\beta+1}{2}}f(s,x)\right|^2\frac{dsdx}{s}$$

$$= C\left(\operatorname{Re}\alpha - \frac{\beta-1}{2}\right)\iint\limits_{\mathbb{R}^{1+n}_+} |f(s,x)|^2 s^\beta ds dx.$$

Hence, $\mathcal{J}_\alpha$ is well defined for $\operatorname{Re}\alpha > \frac{\beta-1}{2}$ and bounded on $T^{2,2,m}_\beta$ for all m. Notice that $\beta < 1$ implies that this domain contains $\alpha = 0$ and $\mathcal{J}_0 = T_2$.

Hence, choose $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geq 0$ such that

$$v(\alpha,q) := \operatorname{Re}\alpha - \frac{\beta-1}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right) > \frac{n}{2m}.$$

This way, in the calculations below we then use the $L^q - L^2$ off-diagonal decay of the kernel K with the following (smaller) order

$$\widehat{M} = \widehat{M}_\alpha = \min\big(M, v(\alpha,q)\big) > \frac{n}{2m}.$$

Now we let f be a $T^{p,2,m}_\beta$ function supported in the Carleson box $\big(0, r_B^m\big] \times B(x_B, r_B)$ and estimate $\mathcal{J}_\alpha(f)$ for $\operatorname{Re}\alpha > \frac{\beta-1}{2}$. We let

$$f_j(t,y) = \begin{cases} \mathcal{J}_\alpha(f)(t,y) & \text{if } 2^j r_B \leq |y - x_B| < 2^{j+1} r_B \quad \text{and} \quad t < 2^{jm} r_B^m, \\ \mathcal{J}_\alpha(f)(t,y) & \text{if } \quad |y - x_B| < 2^{(j+1)} r_B \quad \text{and} \quad 2^{jm} r_B^m \leq t < 2^{(j+1)m} r_B, \\ 0 & \text{otherwise}, \end{cases}$$

for $j \geq 2$ and $f_1(t,y) = \mathcal{J}_\alpha(f)(t,y)$ if $|y - x_B| \leq 2r_B$ and $t < 2^m r_B^m$, 0 elsewhere, so that $\mathcal{J}_\alpha(f) = \sum_{j=1}^\infty f_j$. For $j \geq 2$

$$\int_0^{2^{(j+1)m} r_B^m} \int_{B(x_B, 2^{j+1} r_B)} |f_j(t,y)|^2 t^\beta dt dy = \int_{2^j r_B \leq |y - x_B| < 2^{j+1} r_B} \int_0^{2^{jm} r_B^m} |f_j(t,y)|^2 t^\beta dt dy$$
$$+ \int_{|y - x_B| < 2^{j+1} r_B} \int_{2^{jm} r_B^m}^{2^{(j+1)m} r_B^m} |f_j(t,y)|^2 t^\beta dt dy.$$

Call I_j and J_j the square roots of the first and the second integrals.

For I_j, we split the integral for f_j as $\sum_{k\geq 1} \int_{2^{-k-1}t}^{2^{-k}t}$ so that by Minkowski inequality $I_j \leq \sum_k I_{j,k}$ with

$$I^2_{j,k} = \int_{2^j r_B \leq |y - x_B| < 2^{j+1} r_B} \int_0^{2^{jm} r_B^m} \left| \int_{2^{-k-1}t}^{2^{-k}t} \left(\frac{s}{t}\right)^\alpha \big(K(t,s) f(s,\cdot)\big)(y) ds \right|^2 t^\beta dt dy.$$

Using Cauchy-Schwarz inequality in the s integral and the $L^q - L^2$ off-diagonal decay of order $\widehat{M}$ for the kernel K, with $t \simeq t - s$, we get

$$I^2_{j,k} \lesssim \int_0^{2^{jm} r_B^m} \left(2^{-k} t\right) \int_{2^{-k-1}t}^{2^{-k}t} \left(\frac{s}{t}\right)^{2\operatorname{Re}\alpha} \frac{1}{t^{2 + \frac{2n}{m}\left(\frac{1}{q} - \frac{1}{2}\right)}} \left(1 + \frac{2^{jm} r_B}{t}\right)^{-2\widehat{M}} \|f(s,\cdot)\|_q^2 ds t^\beta dt$$

$$\lesssim 2^{-2m\widehat{M}j} r_B^{-2\widehat{M}} \int_0^{2^{jm}r_B^m} \left(2^{-k}t\right) \int_{2^{-k-1}t}^{2^{-k}t} \left(2^{-k}\right)^{2\mathrm{Re}\,\alpha} \frac{1}{t^{2+\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)-2\widehat{M}}} \|f(s,\cdot)\|_q^2 ds t^{\beta} dt$$

$$\lesssim 2^{-2m\widehat{M}j} |B|^{-2\frac{\widehat{M}}{n}} 2^{k(-2\mathrm{Re}\,\alpha+\beta-1)} \int_0^{2^{jm-k}r_B^m} \|f(s,\cdot)\|_q^2 s^{\beta} \left(2^k s\right)^{2\widehat{M}-\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)} ds.$$

Now, by letting

$$2\widehat{M} - \frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right) = \frac{2n}{m}\left(\frac{1}{p_M^{\alpha}} - \frac{1}{q}\right),$$

using the support condition on f that $s \le r_B^m$ and the norm estimate on f via Corollary 2.2.7 (which extends to general m and β), together with $\widehat{M} > \frac{n}{2m} \ge \frac{n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)$, we have

$$I_{j,k}^2 \lesssim 2^{-2m\widehat{M}j} |B|^{-2\frac{\widehat{M}}{n}} 2^{k(-2\mathrm{Re}\,\alpha+\beta-1)} 2^{\inf(k,jm)\left[2\widehat{M}-\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)\right]} \|f\|_{T_\beta^{p_M^{\alpha},2,m}}.$$

Hence we obtain

$$\sum_{k\ge 1} I_{j,k} \lesssim jm 2^{-m\widehat{M}j} r_B^{-\widehat{M}} \|f\|_{T_\beta^{p_M^{\alpha},2,m}}.$$

For J_j, we remark that the support of f forces $s \le r_B^m$ while $t \simeq 2^{jm} r_B^m \ge 2 r_B^m$. Hence

$$J_j^2 \lesssim \int_{|y-x_B|\le 2^j r_B} \int_{2^{jm}r_B^m}^{2^{(j+1)m}r_B^m} \int_0^{r_B^m} \left(\frac{s}{t}\right)^{2\mathrm{Re}\,\alpha-2\frac{\beta-1}{2}} \left|\left(tK(t,s)s^{\frac{\beta+1}{2}} f(s,\cdot)\right)(y)\right|^2 \frac{ds}{s}\frac{dt}{t} dy$$

$$\lesssim \int_{2^{jm}r_B^m}^{2^{(j+1)m}r_B^m} \int_0^{r_B^m} \left(\frac{s}{t}\right)^{2\mathrm{Re}\,\alpha-2\frac{\beta-1}{2}} \frac{1}{t^{\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)}} \left\|s^{\frac{\beta+1}{2}} f(s,\cdot)\right\|_q^2 \frac{ds}{s}\frac{dt}{t}$$

$$\lesssim 2^{-jm\left(2\left(\mathrm{Re}\,\alpha-\frac{\beta-1}{2}\right)+\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)\right)} r_B^{-m\left(2\left(\mathrm{Re}\,\alpha-\frac{\beta-1}{2}\right)+\frac{2n}{m}\left(\frac{1}{q}-\frac{1}{2}\right)\right)} \|f\|^2_{T_\beta^{\widetilde{p}_c^{\alpha},2,m}}$$

$$\lesssim 2^{-2mv(\alpha,q)j} |B|^{-2\frac{v(\alpha,q)}{n}} \|f\|^2_{T_\beta^{\widetilde{p}_c^{\alpha},2,m}}.$$

The exponent $\widetilde{p}_c^{\alpha}$ is such that

$$v(\alpha,q) = \frac{n}{m}\left(\frac{1}{\widetilde{p}_c^{\alpha}} - \frac{1}{2}\right).$$

In the next to last estimate we used the norm estimate on f, with Corollary 2.2.7 (which extends to general m and β). Note that we used the relation

$$\frac{n}{m}\left(\frac{1}{\widetilde{p}_c^{\alpha}} - \frac{1}{q}\right) = \mathrm{Re}\,\alpha - \frac{\beta-1}{2}.$$

in applying Corollary 2.2.7.

In all, we have

$$\begin{aligned}
&\left(\int_0^{2^{(j+1)m}r_B^m}\int_{B(x_B,2^{j+1}r_B)}|f_j(t,y)|^2t^\beta\,dt\,dy\right)^{1/2}\\
&\quad\lesssim \max\left(jm2^{-m\widehat{M}j}|B|^{-\frac{\widehat{M}}{n}}\|f\|_{T_\beta^{p_M^\alpha,2,m}},\,2^{-mv(\alpha,q)j}|B|^{-\frac{v(\alpha,q)}{n}}\|f\|_{T_\beta^{\widetilde{p}_c^\alpha,2,m}}\right)\\
&\quad\lesssim jm2^{-m\widehat{M}j}|B|^{-\frac{\widehat{M}}{n}}\|f\|_{T_\beta^{p_M^\alpha,2,m}}.
\end{aligned}$$

Recall that $\widehat{M}$ is strictly larger than $\frac{n}{2m}$.

We now start the discussion. Case (2) of Theorem 2.1.6 corresponds to $v(0,q)>\frac{n}{2m}$. Applying Theorem 2.2.1 (note that the factor jm is harmless), $\mathscr{J}_0$ extends to a bounded operator mapping $T_\beta^{\widehat{p}_c,2,m}$ into ${}^wT_\beta^{\widehat{p}_c,2,m}$, with $\widehat{p}_c<1$ given by

$$\frac{n}{m}\left(\frac{1}{\widehat{p}_c}-\frac{1}{2}\right)=\widehat{M}=\min\big(M,v(0,q)\big),$$

which means $\widehat{p}_c=\max\big(p_M,\widetilde{p}_c\big)$, where $\widetilde{p}_c$ is as in the statement of Theorem 2.1.6. By Marcinkiewicz interpolation with $p=2$ result, $\mathscr{J}_0$ extends to a bounded map on $T_\beta^{p,2,m}$ for $\max\big(p_M,\widetilde{p}_c\big)<p\le 2$.

Case (1) of Theorem 2.1.6 corresponds to $v(0,q)\le\frac{n}{2m}$. Let $\alpha_1>0$ be such that $v(\alpha_1,q)=\frac{n}{2m}$. As in the preceding case, for any $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>\alpha_1$, by Theorem 2.2.1 (and Marcinkiewicz interpolation) $\mathscr{J}_\alpha$ extends to a bounded map on $T_\beta^{1,2,m}$, $\beta<1$, and by checking the proof above, the bounds does not depend on $\operatorname{Im}\alpha$. By the $p=2$ case, if $\alpha_2=\frac{\beta-1}{2}>0$, for any $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>\alpha_2$, $\mathscr{J}_\alpha$ extends to a bounded map on $T_\beta^{2,2,m}$ and the bounds does not depend on $\operatorname{Im}\alpha$. Hence, by Stein's interpolation theorem for analytic families extended to tent spaces (see [HTV91] for its extension to $T^{p,2}$ with $p\ge1$ and apply the isometry in Remark 2.1.2 for the general tent spaces $T_\beta^{p,2,m}$), $\mathscr{J}_0$ extends to a bounded map on $T_\beta^{p,2,m}$ for $p_c<p\le2$ and p_c is the exponent with

$$\frac{1}{p_c}=\frac{\theta}{1}+\frac{1-\theta}{2},\quad\text{when } 1=\theta\alpha_1+(1-\theta)\alpha_2.$$

A calculation yields the right value of p_c in the statement of Theorem 2.1.6.

2.5 Remarks

We give some remarks to end this chapter.

(R1). At this moment, for Theorems 2.2.1–2.2.2 in concrete settings, say for the maximal regularity operators to be studied in Part II, we do not know good examples of regularization families $\{A_{r_B}\}$ adapted to the maximal regularity operator in question.

Here *adapted* means that the regularization operator $\{A_{r_B}\}$ has to take care of the regular part T_2 of $T \in SIO^+$ in some proper manner. Note that the part T_2 is most involved in our calculations above. We mention that in the study of $L^p\left(\mathbb{R}^{1+n}_+\right)$-boundedness, instead of the tent space boundedness here, of maximal regularity operators, there does exist a good choice of regularization families (see [BZ09] for further information).

(R2). The claim in part (1) of Theorem 2.1.6 does not give a *desired* extrapolation range, as compared with the one in part (2) and as seen in Part II of this thesis.

(R3). For the *lack* of endpoint weak type tent space estimates in part (1) of Theorem 2.1.6, it is, also in its own interest, meaningful to know if Stein's interpolation [Ste56] for analytic families of operators holds on quasi-Banach and Lorentz type tent spaces. In the Lebesgue- and Hardy-space settings, such results exist. See for example, [Sag69] on Lorentz type and [CS88] on quasi-Banach, extensions of Stein's interpolation.

(R4). In view of Section 2.3, as the final remark (which is also the most exotic one) we wonder whether there exists an *effective*[5] tent-space Calderón-Zygmund theory on $T^{p,2}$ for $1 < p < 2$. This may be an *ill-posed* question, as observed in [AKMP12].

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5. To make it a little bit more precise, by this effectiveness I mean a Calderón-Zygmund theory which does not have to go down by analytic interpolation into the atomic case ($p < 1$).

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Part II

MAXIMAL REGULARITY OPERATORS —— STABILITY OF R-ANALYTICITY

3

Maximal regularity in tent spaces and improved Blunck-Kunstmann criteria for the extrapolation of R-analyticity

Abstract

We give a sufficient condition for the maximal regularity in tent spaces, namely, the tent space boundedness of maximal regularity operators.

In particular, for a $2m$-order complex-valued (not necessarily divergence form) elliptic operator L on $\mathbb{R}^n$, with m and n two integers not less than 1, we prove in this chapter that the L-associated forward maximal regularity operator $\mathbf{M}_L^+$ extends to a bounded operator on the parabolic tent space $T^{p,2,2m}((0,\infty)\times\mathbb{R}^n)$ for $(p_-)_* := \frac{np_-}{n+mp_-} < p \le \infty$, with $p_- \in [1,2)$ being the infimum of q for which the complex analytic semigroup $\{e^{-zL}\}_{z\in S_\delta}$, $\delta \in (0,\pi/2)$, where $S_\delta = \{te^{i\theta} : t>0, |\arg\theta| < \delta\}$, satisfies certain large order $L^q(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal decay.

The interesting part lies in the tent space boundedness for $p_- < p < 2$, and our machinery is particularly designed to take care of the case $(p_-)' \le \frac{n}{m}$, where $\frac{1}{p_-} + \frac{1}{(p_-)'} = 1$. This thereby complements the recent extrapolation theory of Auscher et al. for the maximal regularity operators on tent spaces.

Our requirements on the $L^q(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal decay of the analytic semigroup are weaker than those needed by the extrapolation criteria of Blunck-Kunstmann and Kunstmann for the analytic generator's R-analyticity in $L^p(\mathbb{R}^n)$ and maximal $L^p(\mathbb{R}^n)$-regularity for $q < p < 2$.

Contents

3.1 Introduction

Let $\mathbb{R}^{1+n}_+$ be the upper half-space $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0,\infty)$. Let $m, n \in \mathbb{N}_+$, the set of integers not less than 1. Let $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+\right)$ be the class of locally square integrable functions on $\mathbb{R}^{1+n}_+$, and for $0 < p < \infty$, let $L^p = L^p\left(\mathbb{R}^n\right)$ be the class of Lebesgue p-integrable functions on $\mathbb{R}^n$, with its quasi-norm simply denoted by $\|\cdot\|_p$. Let $|\cdot|$ be the Euclidean distance or volume of sets in $\mathbb{R}^n$, or, the modulus of real and complex scalars.

Definition 3.1.1. Define the tent space $T^{p,2,m}(dtdy)$, $0 < p < \infty$, as the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+\right)$ functions such that

$$\|F\|_{T^{p,2,m}(dtdy)} := \left(\int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,t^{1/m})}(y)}{t^{n/m}} |F(t,y)|^2 dtdy \right)^{p/2} dx \right)^{1/p} < \infty.$$

Define $T^{\infty,2,m}(dtdy)$ as the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+\right)$ functions such that

$$\|F\|_{T^{\infty,2,m}(dtdy)} := \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \left(r^{-n} \int_0^{r^m} \int_{B(x,r)} |F(t,y)|^2 dtdy \right)^{1/2} < \infty.$$

The scale of classical tent spaces T^p_2, $0 < p \le \infty$, which corresponds to $T^{p,2,m}(dtdy)$ in the elliptic [1] setting, was introduced in [CMS85] as an intertwining tool for many topics around real variable harmonic analysis.

Remark 3.1.2. Recall that the map $\iota: T^{p,2,m}(dtdy) \to T^p_2$ defined by

$$\iota(F)(t,y) := \sqrt{m}\, t^{\frac{m}{2}} F(t^m, y) \tag{3.1.1}$$

is an isometry.

Let Λ be an $L^2\left(\mathbb{R}^n\right)$-sectorial operator, that is, $-\Lambda$ is a densely defined closed linear operator acting on L^2 and generating a bounded analytic semigroup $\left\{e^{-t\Lambda}\right\}_{t\ge 0}$. Consider the L-associated forward maximal regularity operator

$$\mathbf{M}^+_\Lambda : F \mapsto \mathbf{M}^+_\Lambda(F), \quad \mathbf{M}^+_\Lambda(F)_t := \int_0^t \Lambda e^{-(t-s)\Lambda} F_s\, ds, \tag{3.1.2}$$

1. Namely, in the above definition of $T^{p,2,m}(dtdy)$, one replaces $dtdy$ by $t^{-1}dtdy$ and set $m = 1$.

originally defined on $F \in L^2(\mathbb{R}_+, dt; \mathbf{D}(\Lambda))$. Here and below, for a function F defined on $\mathbb{R}_+^{1+n}$, we let $F_s = F(s,\cdot)$. By a classical result of L. de Simon [dS64], $\mathbf{M}_\Lambda^+$ extends to a bounded operator on $L^2\left(\mathbb{R}_+, dt; L^2(\mathbb{R}^n)\right)$. By Fubini's theorem

$$T^{2,2,m}\left(\mathbb{R}_+^{1+n}\right) \simeq L^2\left(\mathbb{R}_+^{1+n}\right) = L^2\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right) = L^2\left(\mathbb{R}_+; L^2\left(\mathbb{R}^n\right)\right)$$

with equivalence of norms for whatever m.

The maximal regularity operator is a typical example of singular integral operators with operator valued kernels. We recall the following measure of decay and hypercontractivity on such kernels.

Definition 3.1.3. Let $\langle a/b\rangle_m := (1 + a^m/b)$. Let $\operatorname{dist}(E,F) := \inf\{|x-y| \mid x \in E, y \in F\}$. Let $1 \le q \le 2 \le r \le \infty$. For $0 < \delta < \pi/2$, let $S_\delta := \left\{te^{i\theta} : t > 0, |\arg\theta| < \delta\right\}$.

A class of uniformly $L^2(\mathbb{R}^n)$ bounded operators $\{T(t)\}_{t>0}$ is said to satisfy the $L^q - L^r$ off-diagonal decay, with homogeneity m and with decay order M, if

$$\left\|\mathbf{1}_{B_2} T(t) \mathbf{1}_{B_1} f\right\|_r \lesssim t^{-\frac{n}{m}\left(\frac{1}{q}-\frac{1}{r}\right)} \langle \operatorname{dist}(B_1, B_2)/t\rangle_m^{-M} \left\|\mathbf{1}_{B_1} f\right\|_q$$

for all balls $B_1, B_2 \subset \mathbb{R}^n$, all $t > 0$ and all $f \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

A class of uniformly $L^2(\mathbb{R}^n)$ bounded operators $\{T(z)\}_{z\in S_\delta}$, $0 < \delta < \pi/2$, is said to satisfy the $L^q - L^r$ off-diagonal decay, with homogeneity m and with decay order M, if

$$\left\|\mathbf{1}_{B_2} T(z) \mathbf{1}_{B_1} f\right\|_r \lesssim |z|^{-\frac{n}{m}\left(\frac{1}{q}-\frac{1}{r}\right)} \langle \operatorname{dist}(B_1, B_2)/|z|\rangle_m^{-M} \left\|\mathbf{1}_{B_1} f\right\|_q$$

for all balls $B_1, B_2 \subset \mathbb{R}^n$, all $z \in S_\delta$ and all $f \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

An operator family is said to satisfy the $L^q - L^r$ off-diagonal decay with homogeneity m, if it satisfies the $L^q - L^r$ off-diagonal decay with homogeneity m for any $M > 0$.

As seen in [AMR08, HvNP08], the above notion (for $q = r = 2$) on the time-space off-diagonal decay is pertinent to the extrapolation problems of integrals operators on tent spaces. This certainly includes the extrapolation of the maximal regularity operators on tent spaces. We shall refer to the tent space boundedness of maximal regularity operators as Conical Maximal Regularity (CMR).

Theorem 3.1.4 ([AMP12])**.** *Let $p \in \left(\frac{2n}{n+2m}, \infty\right] \cap (1,\infty]$ and $\tau = \min(p,2)$. If $\left\{t\Lambda e^{-t\Lambda}\right\}_{t>0}$ satisfies the $L^2 - L^2$ off-diagonal decay with homogeneity m and with decay order $M > \frac{n}{m\tau}$, then $\mathbf{M}_\Lambda^+$ extends to a bounded operator on $T^{p,2,m}(dt\,dy)$.*

The proof of this theorem uses the change of aperture results in tent spaces. For $x \in \mathbb{R}^n$ and $a > 0$, we define

$$\mathscr{A}^a(F)(x) := \left(\iint_{\mathbb{R}_+^{1+n}} \frac{\mathbf{1}_{B(x,at^{1/m})}(y)}{t^{n/m}} |F(t,y)|^2 \, dt\, dy\right)^{1/2},$$

and we omit a if $a = 1$. Hence via $\mathscr{A}^a$ one can also define a scale of tent spaces. The change of aperture in tent spaces amounts to say the equivalence of tent space quasi-norms for different apertures a. The sharp dependence in a is obtained in [Aus11] by using atomic decompositions of tent spaces. See also [FS72, Tor86, Uch01, HvNP08].

In this chapter we shall use the following result.

Remark 3.1.5 (Change of Apertures)**.** For any $a \geq 1$ and $0 < p \leq 2$

$$\left\| \mathscr{A}^a(F) \right\|_{L^p} \leq C(n,m,p)\, a^{\frac{n}{p}} \| \mathscr{A}(F) \|_{L^p}.$$

To see this it suffices to use the isometry (3.1.1) and apply results of [Aus11] on T_2^p.

Inspired by [Aus11], P. Auscher et al. then use the atomic decompositions of tent spaces to improve Theorem 3.1.4 as follows.

Theorem 3.1.6 ([AKMP12])**.** *Let $1 \leq q \leq 2$. Let $M > \frac{n}{2m}$. Let $p_M < 1$ be given by $M = \frac{n}{m}\left(\frac{1}{p_M} - \frac{1}{2}\right)$. If $\left\{t\Lambda e^{-t\Lambda}\right\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order M, then $\mathbf{M}_\Lambda^+$ extends to a bounded operator on $T^{p,2,m}(dt dy)$ for*

A) $p_c < p \leq \infty$, where $p_c := \frac{4n}{2n+mq'} \geq 1$, if $mq' \leq 2n$;

B) $\max\left(p_M, q_*\right) < p \leq \infty$, where $q_* := \frac{2nq}{2n+mq} < 1$, if $mq' > 2n$.

Here as usual, q' is given by $\frac{1}{q} + \frac{1}{q'} = 1$.

We stick to the use of $L^q - L^2$off-diagonal decay but with the threshold order[2] $M > \frac{n}{mq}$ as required in Theorem 3.1.4. We aim, with this slightly stronger condition, to eliminate the case A) in Theorem 3.1.6, and arrive at a boundedness result on $T^{p,2,m}(dt dy)$ for $p > q_*$ as in the case B), even for a $q < 2$ very closed[3] to 2.

More precisely, we shall prove

Theorem 3.1.7. *Let $1 \leq q < 2$ and assume $mq' \leq 2n$ so that $q_* = \frac{2nq}{2n+mq} \geq 1$. Let $\left\{e^{-t\Lambda}\right\}_{t>0}$ be a bounded real semigroup in $L^2(\mathbb{R}^n)$.*

I) Assume that the differentiated family $\left\{t\Lambda e^{-t\Lambda}\right\}_{t>0}$ satisfies the $L^2 - L^2$ off-diagonal decay with homogeneity m. Assume that the real semigroup $\left\{e^{-t\Lambda}\right\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m. Then $\mathbf{M}_\Lambda^+$, originally defined as in (3.1.2), extends to a bounded operator on $T^{p,2,m}(dt dy)$ for $q_* < p \leq \infty$.

II) Assume that the complex analytic semigroup $\left\{e^{-z\Lambda}\right\}_{z\in S_\delta}$ for some $0 < \delta < \pi/2$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m. Then $\mathbf{M}_\Lambda^+$, originally defined as in (3.1.2), extends to a bounded operator on $T^{p,2,m}(dt dy)$ for $q_* < p \leq \infty$.

A quick remark about the extrapolation exponents first. Let $\theta = \theta_q = \frac{mq'}{2n}$ so that $0 < \theta \leq 1$. We shall obtain $p = q_*$ from the interpolation relation

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{q},$$

2. We do not precise the threshold in stating Theorem 3.1.7. But one infers this from its proof.
3. But we also note that the case when $q = 2$ is already covered by Theorem 3.1.6, and $p_c = 2_*$.

and we can obtain $p = p_c$ (by calculations as suggested in [AKMP12]) from

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}.$$

Both calculations follows by the standard Stein's analytic interpolation process, from which θ is determined. We see that $p_c \geq q_*$.

Hence our result complements case B) of Theorem 3.1.6 when $mq' \leq 2n$. Note that compared to Theorem 3.1.4 and Theorem 3.1.6, we also imposed the off-diagonal decay conditions on the semigroup $\{e^{-t\Lambda}\}_{t>0}$. We also note that Part II) of the above theorem follows from Part I). This is due to the use of Hölder's inequality to have $L^2 - L^2$ off-diagonal decay for $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ from its $L^q - L^2$ off-diagonal decay (see [AM07]), and the use of Cauchy formula to have the off-diagonal decay for $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ from the one on $\{e^{-z\Lambda}\}_{z\in S_\delta}$ for some $0<\delta<\pi/2$.

As a consequence of the general extrapolation results in Theorem 3.1.7, we have the following maximal regularity in parabolic tent spaces. We shall give in Section 3.4 more details about the class of elliptic operators involved in next theorem.

Theorem 3.1.8. *Let $m \in \mathbb{N}_+$. Let L be an an elliptic divergence operator of order m as in (3.4.1) or an elliptic non-divergence operator of order $2m$ as in (3.4.2). Let p_- be the infimum of q for which $-L$ generates a bounded analytic semigroup on $L^q(\mathbb{R}^n)$. Then the maximal regularity operator*

$$\mathbf{M}_L^+ : F \mapsto \mathbf{M}_L^+(F), \quad \mathbf{M}_L^+(F)_t := \int_0^t L e^{-(t-s)L} F_s\, ds,$$

originally defined on $F \in L^2(\mathbb{R}_+, dt; \mathbf{D}(L))$, extends to a bounded operator on the parabolic tent space $T^{p,2,2m}(dt\,dy)$ for $\frac{np_-}{n+mp_-} < p \leq \infty$.

For the convenience of the reader, we point out the change of homogeneity (from m to $2m$) in the above theorem and its proof.

Our arguments in proving Theorem 3.1.7 (see Lemma 3.2.3 below) yield the following improvement on the Blunck-Kunstmann criteria for R-boundedness in [BK02], and in particular, on the criteria for R-analyticity. As a well-known fact, this also has impact on the maximal $L^p(\mathbb{R}^n)$-regularity, $1<p<\infty$.

Theorem 3.1.9. *Let $1 \leq q < 2$. Assume that the uniformly $L^2(\mathbb{R}^n)$ bounded operator family $\mathscr{T} = \{T(t)\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order $M^{\mathscr{T}} > \frac{n}{mq}$. Then $\{T(t)\}_{t>0}$ satisfies*

$$\left\| \left(\int_0^\infty |T(t)F_t(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \lesssim \left\| \left(\int_0^\infty |F_t(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_p$$

for any $q < p \leq 2$.

The inequality in Theorem 3.1.9 is known to be equivalent to the R-boundedness of $\mathscr{T}$ in $L^p(\mathbb{R}^n)$. The uniform $L^2(\mathbb{R}^n)$ boundedness of $\mathscr{T}$ implies its R-boundedness in $L^2(\mathbb{R}^n)$. Our extrapolation criteria for R-boundedness improves, at the level of the threshold of off-diagonal decay, those in [BK02] and [Kun08]. The precise comparison will be given in subsection 3.5.2.

In particular, assume that both the real semigroup $\{e^{-t\Lambda}\}_{t>0}$ and its differentiated family $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ satisfy $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order $M > \frac{n}{mq}$, or assume that the analytic semigroup $\{e^{-z\Lambda}\}_{z\in S_\delta}$ for some $0 < \delta < \pi/2$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order $M > \frac{n}{mq}$, then by Theorem 3.1.9 and for $q < p \le 2$, we have

the R-boundedness of both the real semigroup $\{e^{-t\Lambda}\}_{t>0}$ and its differentiated family $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ in $L^p(\mathbb{R}^n)$,

which is often referred to as R-analyticity of the analytic generator $-\Lambda$ (see [KW04b, Theorem 1.11]). By the famous characterization of L. Weis [Wei01], R-analyticity of the generator $-\Lambda$ in $L^p(\mathbb{R}^n)$ is equivalent to the maximal $L^p(\mathbb{R}^n)$-regularity for $-\Lambda$.

Some more words on the assumptions of Theorem 3.1.7. First, we point out that the *a priori*[4] analyticity of the semigroup is in $L^2(\mathbb{R}^n)$, which is equivalent to the R-analyticity in $L^2(\mathbb{R}^n)$. Given the assumptions in part II), namely, the $L^q - L^2$ off-diagonal decay of the complex analytic semigroup, we deduce its analyticity in $L^q(\mathbb{R}^n)$. This involves an argument from [Aus07, Lemma 3.3], provided that the order M for $L^q - L^2$ off-diagonal decay of the semigroup (in the case $m = 1$ in [Aus07]) satisfies $M > \frac{n}{q} - \frac{n}{2}$, which is always valid in our setting. Next, for the extrapolation theory on tent spaces, the threshold for the off-diagonal decay order on the differentiated family $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ can be lowered to $M^* > \frac{n}{mq_*} = \frac{n}{mq} + \frac{1}{2}$ by using the change of apertures as in [AMP12], or even to $M^* > \frac{n}{2m}$ by using the atomic decompositions of tent spaces as in [AKMP12]. We choose not to precise this threshold for two reasons:

1) We always need the large order $L^q - L^2$ off-diagonal decay on the semigroup $\{e^{-t\Lambda}\}_{t>0}$, hence it is less interesting to lower the decay order for $\{t\Lambda e^{-t\Lambda}\}_{t>0}$;

2) We only consider the tent space boundedness for maximal regularity operators associated to elliptic operators L as formulated in Theorem 3.1.8, and usually, the decay order of $\{tLe^{-tL}\}_{t>0}$ can be taken as large as we want.

The large order off-diagonal estimates are also used to extrapolate the bounded holomorphic functional calculus, see [BK03]. However, this part of information is not needed here. In contrast with this, Chapter 5 on conical maximal regularity for perturbed first order Dirac operator relies heavily on functional calculus. Hence we shall address these issues in Chapter 5.

4. This is also prescribed into the $L^q - L^2$ off-diagonal decay condition.

3.2 Main tools on R-analyticity

We state a series of lemmata as our main tools in proving Theorem 3.1.7 and Theorem 3.1.9. Recall that a generator $-\Lambda$ of a bounded analytic semigroup $\{e^{-t\Lambda}\}_{t>0}$ on L^p is R-analytic if both $\{e^{-t\Lambda}\}_{t>0}$ and $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ are R-bounded in L^p.

3.2.1 Vertical Maximal Regularity (VMR)

We shall need the following result proved in [vNVW15].

Lemma 3.2.1 (Vertical Maximal Regularity)**.** *If* $-\Lambda$ *is R-analytic in* L^p*,* $1<p<\infty$*, then*

$$\left\|\mathbf{M}_{\Lambda}^{+}(F)\right\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+))} \lesssim \|F\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+))}. \tag{3.2.1}$$

The proof of this result uses the Kalton-Weis γ-multiplier theorem [KW04a, vN10]. Indeed, in the Banach space $L^p(\mathbb{R}^n)$, $1<p<\infty$, the vertical maximal regularity (3.2.1) is equivalent to

$$\left\|\mathbf{M}_{\Lambda}^{+}(F)\right\|_{L^2(\mathbb{R}_+;L^p(\mathbb{R}^n))} \lesssim \|F\|_{L^2(\mathbb{R}_+;L^p(\mathbb{R}^n))}. \tag{3.2.2}$$

The inequality (3.2.2) is often referred to as maximal $L^p(\mathbb{R}^n)$-regularity of $-\Lambda$. That R-analyticity of $-\Lambda$ in L^p, $1<p<\infty$, implies its maximal $L^p(\mathbb{R}^n)$-regularity is a classical result proved in [Wei01, Corollary 4.4].

3.2.2 R-boundedness and Schur estimates

Given an operator-valued kernel $\{K(t,s)\}_{0<s<t<\infty}$ and $z\in\mathbb{C}$ with $\operatorname{Re} z>0$, consider

$$K_z^{+}(t,s) = \left[\mathbf{1}_{\mathbb{R}_+}(t-s)\right]\left(\frac{s}{t}\right)^{z} K(t,s).$$

We have the following vector-valued boundedness result.

Lemma 3.2.2. *Suppose* $\{K(t,s)\}_{s\neq t}$ *is R-bounded in* L^p*. For* $z\in\mathbb{C}$ *with* $\operatorname{Re} z>0$*, consider*

$$T_{K_z}(F)_t = \int_0^t K_z(t,s)F_s\frac{ds}{s},$$

defined for $F\in V^p = L^p\left(\mathbb{R}^n;L^2(\mathbb{R}_+,t^{-1}dt)\right)$*. Then*

$$\left\|T_{K_z}\right\|_{V^p\to V^p} \lesssim e^{|\operatorname{Im} z|},$$

where the implicit constant may depend on $\operatorname{Re} z$*, but not on* $\operatorname{Im} z$*.*

Proof. This is proved in [FMP14] for real z. The extension to complex z is straightforward, with the right dependence on z as in the statement. The proof also shows that $T_{K_z}(F)_t$ is well-defined by the above integral. □

3.2.3 Reverse Hölder Inequalities (RHI)

We turn to the third tool that we shall use. It is shown (see for example [AHM12]) that $T^{p,2,m}(dtdy) \hookrightarrow L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right)$ for $0 < p < 2$, and the inclusion is strict.

Lemma 3.2.3 (Reverse Hölder Inequalities)**.** *Let $1 \le q < 2$. Suppose that the ($L^2(\mathbb{R}^n)$-bounded) semigroup $\{e^{-t\Lambda}\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order $M > \frac{n}{mq}$. For $F \in L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right)$ and $t > 0$, set*

$$\mathbf{G}^+_\Lambda(F)_t = \fint_{[t/4,t/2]} e^{-(t-s)\Lambda} F_s \, ds. \tag{3.2.3}$$

Then one has the estimate

$$\left\| \mathbf{G}^+_\Lambda(F) \right\|_{T^{p,2,m}(dtdy)} \lesssim \|F\|_{L^p(\mathbb{R}^n; L^2(\mathbb{R}_+))} \tag{3.2.4}$$

for $p \in (q, 2]$.

We call this lemma Reverse Hölder Inequalities since we have a similar (local) regularization effect in the $L^q - L^2$ off-diagonal decay of the semigroup $\{e^{-t\Lambda}\}_{t>0}$. We prove this lemma later in Subsection 3.3.4.

3.3 Proof of Theorem 3.1.7

First we remark that results in the less technical case $2 < p < \infty$ follows immediately by interpolation between the $p = 2$ result of L. de Simon [dS64], using the (R)-analyticity in $L^2(\mathbb{R}^n)$ of $-\Lambda$, and the endpoint $p = \infty$ result established in [AMP12, Theorem 3.2], using the $L^2 - L^2$ off-diagonal decay of $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ with homogeneity m and with decay order $M > \frac{n}{2m}$.

Hence it suffices to prove Theorem 3.1.7 for the case $\frac{2nq}{2n+mq} < p < 2$. We provide two different arguments in next two subsections.

3.3.1 First approach: change of apertures and (VMR)

Here we prove Theorem 3.1.7 in the case $q < p < 2$ by using the vertical maximal regularity result given in Lemma 3.2.1, together with the tent space extrapolation theory via change of aperture methods. By (3.2.1) one has

$$T^{p,2,m}(dtdy) \hookrightarrow L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right) \xrightarrow{\mathbf{M}^+_\Lambda} L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right). \tag{3.3.1}$$

The first embedding uses [AHM12, Proposition 2.1] which extends to our setting with slight modifications. Hence $\mathbf{M}^+_\Lambda(F) \in L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right)$ if $F \in T^{p,2,m}(dtdy)$. To show that

$\mathbf{M}_\Lambda^+(F)$ is actually in $T^{p,2,m}(dtdy)$, we set

$$\begin{aligned}\widetilde{\mathbf{M}}_\Lambda^+(F)_t &:= \mathbf{M}_\Lambda^+(F)_t - \fint_{[t/4,t/2]} e^{-(t-s)\Lambda}\mathbf{M}_\Lambda^+(F)_s\,ds \\ &= \mathbf{M}_\Lambda^+(F)_t - \mathbf{G}_\Lambda^+\left(\mathbf{M}_\Lambda^+(F)\right)_t,\end{aligned} \tag{3.3.2}$$

where $F \in T^{p,2,m}(dtdy)$, $q < p < 2$. This decomposition borrows an idea from [AF14]. Note that at least we can interpret the splitting in (3.3.2) as an almost everywhere equality in $L^p(\mathbb{R}^n)$. Indeed, for $p < 2$ there is a natural embedding

$$F \in T^{p,2,m}(dtdy) \hookrightarrow L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right) \hookrightarrow L^2\left(\mathbb{R}_+; L^p\left(\mathbb{R}^n\right)\right),$$

where the first embedding is again by [AHM12] and the second is by Minkowski's integral inequality. By classical maximal regularity, $\mathbf{M}_\Lambda^+(F) \in L^2(\mathbb{R}_+; L^p(\mathbb{R}^n))$ when $L^2(\mathbb{R}_+; L^p(\mathbb{R}^n))$. Moreover, by L^p boundedness of $e^{-(t-s)\Lambda}$, with $s \in [t/4, t/2]$, we also have $\mathbf{G}_\Lambda^+\left(\mathbf{M}_\Lambda^+(F)\right)_t \in L^p$ for each $t > 0$. This justifies the sense of (3.3.2).

According to Lemma 3.2.3 and (3.3.1), we have for the last term in (3.3.2)

$$\left\|\mathbf{G}_\Lambda^+\left(\mathbf{M}_\Lambda^+(F)\right)\right\|_{T^{p,2,m}(dtdy)} \lesssim \left\|\mathbf{M}_\Lambda^+(F)\right\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+))} \lesssim \|F\|_{T^{p,2,m}(dtdy)}.$$

This way, we then see that $\mathbf{M}_\Lambda^+ : T^{p,2,m}(dtdy) \to T^{p,2,m}(dtdy)$ is bounded if and only if $\widetilde{\mathbf{M}}_\Lambda^+ : T^{p,2,m}(dtdy) \to T^{p,2,m}(dtdy)$ is bounded.

To show the boundedness of $\widetilde{\mathbf{M}}_\Lambda^+$ on $T^{p,2,m}(dtdy)$, observe that

$$\widetilde{\mathbf{M}}_\Lambda^+(F)_t = \fint_{[t/4,t/2]} \int_s^t \Lambda e^{-(t-\sigma)\Lambda} F_\sigma\, d\sigma\, ds. \tag{3.3.3}$$

The same reasoning as above for $\mathbf{G}_\Lambda^+\left(\mathbf{M}_\Lambda^+(F)\right)$ shows that for each t, $\widetilde{\mathbf{M}}_\Lambda^+(F)_t$ is a well-defined $L^p(\mathbb{R}^n)$ function. Moreover, for any $t > 0$ we have the time localisation formula

$$\widetilde{\mathbf{M}}_\Lambda^+(F)_t = \widetilde{\mathbf{M}}_\Lambda^+\left(\mathbf{1}_{(t/4,t)}F\right)_t. \tag{3.3.4}$$

Hence for fixed $(t,x) \in \mathbb{R}_+^{1+n}$, with $\mathscr{B}_2(\cdot)(t,x)$ being the L^2 average over $B\left(x, t^{1/m}\right)$,

$$\mathscr{B}_2\left(\widetilde{\mathbf{M}}_\Lambda^+(F)\right)(t,x) = \mathscr{B}_2\left(\widetilde{\mathbf{M}}_\Lambda^+\left(\mathbf{1}_{(t/4,t)}F\right)\right)(t,x).$$

Let $F \in T^{p,2,m}(dtdy)$. Fix $x \in \mathbb{R}^n$. By Minkowski's inequality,

$$\left(\int_0^\infty \left(\mathscr{B}_2\left(\widetilde{\mathbf{M}}_\Lambda^+(F)\right)(t,x)\right)^2 dt\right)^{1/2} \le \sum_{j=0}^\infty \left(\int_0^\infty \left(\mathscr{B}_2\left(\widetilde{\mathbf{M}}_\Lambda^+\left(\mathbf{1}_{S_j(B(x,\sigma^{1/m}))}F(\sigma,\cdot)\right)\right)(t,x)\right)^2 dt\right)^{1/2},$$

where we used the usual annular decomposition of $\mathbb{R}^n$ at level σ and center x

$$S_j\left(B\left(x,\sigma^{1/m}\right)\right) = 2^j B\left(x,\sigma^{1/m}\right) \setminus 2^{j-1}B\left(x,\sigma^{1/m}\right)$$

for $j \ge 1$ and $S_0\left(B\left(x,\sigma^{1/m}\right)\right) = B\left(x,\sigma^{1/m}\right)$. Here, for $\lambda > 0$, $\lambda B(x_B, r_B) = B(x_B, \lambda r_B)$.

Consider first the case $j \le 2$. That $\mathbf{M}_\Lambda^+$ and $\mathbf{G}_\Lambda^+$ are bounded on $L^2(\mathbb{R}_+; L^2(\mathbb{R}^n))$ implies boundedness of $\widetilde{\mathbf{M}}_\Lambda^+$ on $L^2(\mathbb{R}_+; L^2(\mathbb{R}^n))$. Using this and (3.3.4), one obtains

$$\left(t^{-1} \int_t^{2t} \left(\mathscr{B}_2 \left(\widetilde{\mathbf{M}}_\Lambda^+ \left(\mathbf{1}_{B(x,4\sigma^{1/m})} F\right)\right)(s,x)\right)^2 ds \right)^{1/2}$$
$$\lesssim \left| tB\left(x, t^{1/m}\right)\right|^{-1/2} \left\| \widetilde{\mathbf{M}}_\Lambda^+ \left(\mathbf{1}_{B(x,4\sigma^{1/m})} \mathbf{1}_{(t/4,2t)} F\right)\right\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^n))}$$
$$\lesssim \left(t^{-1} \int_{t/4}^{t} \fint_{B(x,4t^{1/m})} |F(s,y)|^2 \, ds dy \right)^{1/2}$$

for each $t > 0$, where, in the last step we used $\sigma < t$ in the representation of $\widetilde{\mathbf{M}}_\Lambda^+$ in (3.3.3). Inserting this estimate into the vertical square integral in t, using Fubini's theorem and change of apertures in tent spaces one further obtains the boundedness

$$\left\| \left(\int_0^\infty \left(\mathscr{B}_2 \left(\widetilde{\mathbf{M}}_\Lambda^+ \left(\mathbf{1}_{B(x,4\sigma^{1/m})} F\right)\right)(s,x)\right)^2 ds \right)^{1/2} \right\|_{L^p} \lesssim \|F\|_{T^{p,2,m}(dtdy)}.$$

Consider now the case $j \ge 3$. Given $(t,x) \in \mathbb{R}_+^{1+n}$, denote

$$F_j(\sigma, y) := F(\sigma, y) \mathbf{1}_{S_j(B(x,\sigma^{1/m}))}(y) \mathbf{1}_{(0,t)}(\sigma), \quad (\sigma, y) \in \mathbb{R}_+^{1+n}.$$

Using Minkowski's inequality, $\frac{t}{4} < s < \sigma < t$ in the representation of $\widetilde{\mathbf{M}}_\Lambda^+$ in (3.3.3), $L^2 - L^2$ off-diagonal estimates for the differentiated family $(t-\sigma)\Lambda e^{-(t-\sigma)\Lambda}$ and Hölder's inequality one obtains, for fixed $(t,x) \in \mathbb{R}_+^{1+n}$,

$$\left\| \widetilde{\mathbf{M}}_\Lambda^+ (F_j)_t \right\|_{L^2(B(x,t^{1/m}))}$$
$$\le \fint_{[t/4,t/2]} \int_s^t (t-\sigma)^{-1} \left\| (t-\sigma)\Lambda e^{-(t-\sigma)\Lambda} F_j(\sigma, \cdot)\right\|_{L^2(B(x,t^{1/m}))} d\sigma ds$$
$$\le \fint_{[t/4,t/2]} \int_s^t (t-\sigma)^{-1} \left\| (t-\sigma)\Lambda e^{-(t-\sigma)\Lambda} F_j(\sigma, \cdot)\right\|_{L^2(B(x,4\sigma^{1/m}))} d\sigma ds$$
$$\lesssim \int_{t/4}^t (t-\sigma)^{-1} \left(\frac{t-\sigma}{2^{mj}\sigma}\right)^N \left\| F_j(\sigma,\cdot)\right\|_{L^2(S_j(B(x,\sigma^{1/m})))} d\sigma$$
$$\lesssim 2^{-mNj} \fint_{[t/4,t]} \left\| F_j(\sigma,\cdot)\right\|_{L^2(S_j(B(x,\sigma^{1/m})))} d\sigma$$
$$\lesssim 2^{-mNj} \left(\fint_{[t/4,t]} \left\| F_j(\sigma,\cdot)\right\|^2_{L^2(S_j(B(x,\sigma^{1/m})))} d\sigma \right)^{1/2}$$
$$\lesssim 2^{-mNj} \left(\fint_{[t/4,t]} \left\| F_j(\sigma,\cdot)\right\|^2_{L^2(S_j(B(x,\sigma^{1/m})))} d\sigma \right)^{1/2}.$$

In the fifth step, we use $N > 0$. Inserting this estimate into the vertical square integral in t, and one can remove the average on $[t/4, t]$, namely, one has

$$\begin{aligned}
&\int_0^\infty \|\widetilde{\mathbf{M}}_\Lambda^+(F_j)_t\|^2_{L^2(B(x,t^{1/m}))} \frac{dt}{t^{n/m}} \\
&\qquad \lesssim 2^{-2mNj} \int_0^\infty \left(\fint_{[t/4,t]} \|F_j(\sigma,\cdot)\|^2_{L^2(S_j(B(x,\sigma^{1/m})))}\, d\sigma \right) \frac{dt}{t^{n/m}} \\
&\qquad \simeq 2^{-2mNj} \int_0^\infty \|F_j(\sigma,\cdot)\|^2_{L^2(S_j(B(x,\sigma^{1/m})))} \frac{d\sigma}{\sigma^{n/m}}.
\end{aligned}$$

Taking an L^p integral in x of the above estimates, one has

$$\|\widetilde{\mathbf{M}}_\Lambda^+(F_j)\|_{T^{p,2,m}(dtdy)} \lesssim 2^{-mNj}(2^j)^{n/p}\|F\|_{T^{p,2,m}(dtdy)}$$

by change of apertures in tent spaces from Lemma 3.1.5. Taking $N > \frac{n}{mq}$ (as we assume N large enough), one can sum over j.

The claim of Theorem 3.1.7 for the case $p > q$ is proved.

3.3.2 Second approach: atomic decompositions and R-boundedness

Here we prove Theorem 3.1.7 in the case $q < p < 2$ by using the R-boundedness of the semigroup $\{e^{-t\Lambda}\}_{t>0}$, together with the tent space extrapolation theory via atomic decomposition methods to precise the off-diagonal decay threshold on $\{t\Lambda e^{-t\Lambda}\}_{t>0}$.

We replace the splitting (3.3.2) by the one used in [AKMP12]. Let

$$\mathbf{M}_\Lambda^{+,2}(F)_t := \int_0^{t/2} \Lambda e^{-(t-s)\Lambda} F_s\, ds,$$

where $F \in T^{p,2,m}(dtdy)$, $q < p < 2$. Let $\mathbf{M}_\Lambda^{+,1} := \mathbf{M}_\Lambda^+ - \mathbf{M}_\Lambda^{+,2}$. Since by our assumption we have for $\{t\Lambda e^{-t\Lambda}\}_{t>0}$ the $L^2 - L^2$ off-diagonal decay with homogeneity m and with order $M > \frac{n}{2m}$, applying Lemma 3.4 of [AKMP12] to $\mathbf{M}_\Lambda^{+,1}$ gives its boundedness on the tent spaces $T^{p,2,m}(dtdy)$ for $p \geq 1$.

For the operator $\mathbf{M}_\Lambda^{+,2}$, we can write

$$\begin{aligned}
\mathbf{M}_\Lambda^{+,2}(F)_t &= \Lambda e^{-\frac{t}{4}\Lambda} \int_0^{t/2} e^{-(\frac{3t}{4}-s)\Lambda} F_s\, ds \\
&= t\Lambda e^{-\frac{t}{8}\Lambda} e^{-\frac{t}{8}\Lambda} \frac{1}{t^{1/2}} \int_0^{t/2} \left(\frac{s}{t}\right)^{1/2} e^{-(\frac{3t}{4}-s)\Lambda} s^{1/2} F_s \frac{ds}{s} \\
&=: t\Lambda e^{-\frac{t}{8}\Lambda} e^{-\frac{t}{8}\Lambda} \frac{1}{t^{1/2}} \mathbf{V}(s^{1/2}F_s)_t,
\end{aligned}$$

where $F \in T^{p,2,m}(dtdy)$, $q < p < 2$. Since by our assumption $\left\{e^{-t\Lambda}\right\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with order $M > \frac{n}{qm}$, we have the R-boundedness of $\left\{e^{-\left(\frac{3t}{4}-s\right)\Lambda}\right\}_{0<s<t/2}$ in $L^p(\mathbb{R}^n)$. Therefore

$$\left\|\mathbf{V}\left(s^{1/2}F_s\right)\right\|_{V^p} \lesssim \left\|s^{1/2}F_s\right\|_{V^p} \lesssim \|F\|_{T^{p,2,m}(dtdy)}.$$

Now using Lemma 3.2.1, we have

$$\begin{aligned}\left\|\mathbf{M}_\Lambda^{+,2}(F)\right\|_{T^{p,2,m}(dtdy)} &\lesssim \left\|e^{-\frac{t}{8}\Lambda}\frac{1}{t^{1/2}}\mathbf{V}\left(s^{1/2}F_s\right)_t\right\|_{T^{p,2,m}(dtdy)} \\ &\lesssim \left\|\frac{1}{t^{1/2}}\mathbf{V}\left(s^{1/2}F_s\right)_t\right\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+))} \\ &= \left\|\mathbf{V}\left(s^{1/2}F_s\right)\right\|_{V^p}.\end{aligned}$$

Here the first inequality follows [5] from the extrapolation theory in [AKMP12] (Note that there is no singularity in the time variable). Hence we proved the theorem.

Remark 3.3.1. This argument lowers the order of $L^2 - L^2$ off-diagonal decay needed on the family $\{t\Lambda e^{-t\Lambda}\}_{t>0}$. But it has two drawbacks. First, it heavily relies on [AKMP12] and is not self-contained. Second, for the potential development of this chapter to the space of homogeneous type in the sense of [CW77], the atomic decomposition of tent spaces would be a very subtle matter.

3.3.3 Analytic interpolation

Hence we prove Theorem 3.1.7 in the case $\frac{2nq}{2n+mq} < p \le q$.

We continue with the atomic decomposition method and use the splitting there. Note that from our assumptions, we have the boundedness of $\mathbf{M}_\Lambda^{+,1}$ on the tent spaces $T^{p,2,m}(dtdy)$ for $p \ge 1$, hence also for $p \ge q_*$ as $q_* = \frac{2nq}{2n+mq} \ge 1$.

It suffices to consider $\mathbf{M}_\Lambda^{+,2}$. We follow a strategy of [AKMP12] and embed $\mathbf{M}_\Lambda^{+,2}$ into the following family of integral operators: for $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -\frac{1}{2}$, define

$$\mathbf{M}_{\Lambda,\alpha}^{+,2}(F)_t := \int_0^{t/2}\left(\frac{s}{t}\right)^\alpha \Lambda e^{-(t-s)\Lambda}F_s\,ds$$

on $F \in T^{p,2,m}(dtdy)$. One can show the boundedness of $\mathbf{M}_{\Lambda,\alpha}^{+,2}$ on the tent spaces $T^{p,2,m}(dtdy)$ for $p > q$ for any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -\frac{1}{2}$. More precisely, one can write

$$\mathbf{M}_{\Lambda,\alpha}^{+,2}(F)_t = \Lambda e^{-\frac{t}{4}\Lambda}\int_0^{t/2}\left(\frac{s}{t}\right)^\alpha e^{-\left(\frac{3t}{4}-s\right)\Lambda}F_s\,ds$$

5. Using the atomic decompositions of tent spaces, the threshold on the order of $L^2 - L^2$ off-diagonal decay is $M > \frac{n}{2m}$ as in [AKMP12], instead of $M > \frac{n}{m}$ by using change of apertures as in [HvNP08]. For a precise statement and its proof in the case $m = 1$, see Theorem 2.3.1 of Chapter 2.

$$= t\Lambda e^{-\frac{t}{8}\Lambda} e^{-\frac{t}{8}\Lambda} \frac{1}{t^{1/2}} \int_0^{t/2} \left(\frac{s}{t}\right)^{\alpha+\frac{1}{2}} e^{-\left(\frac{3t}{4}-s\right)\Lambda} s^{1/2} F_s \frac{ds}{s}$$
$$=: t\Lambda e^{-\frac{t}{8}\Lambda} e^{-\frac{t}{8}\Lambda} \frac{1}{t^{1/2}} \mathbf{V}_\alpha \left(s^{1/2} F_s\right)_t .$$

Then the remaing arguments are similar to those for $\mathbf{M}_\Lambda^{+,2}$. Note that the dependence of the $T^{p,2,m}(dtdy) \to T^{p,2,m}(dtdy)$ operator norm of $\mathbf{M}_{\Lambda,\alpha}^{+,2}$ in α not exceeds $e^{|\mathrm{Im}\,\alpha|}$.

Moreover, applying Lemma 3.5 of [AKMP12], we see that for

$$\mathrm{Re}\,\alpha > \frac{n}{2m} - \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right) - \frac{1}{2} = \frac{n}{q'm} - \frac{1}{2},$$

$\mathbf{M}_{\Lambda,\alpha}^{+,2}$ is bounded on the tent spaces $T^{p,2,m}(dtdy)$ for $p \geq 1$. Precisely, since the order of off-diagonal at hand is large enough, Lemma 3.5 of [AKMP12] gives a criteria for $T^{p,2,m}(t^\beta dtdy)$ boundedness of $\mathbf{M}_{\Lambda,\alpha}^{+,2}$ for $p_c^\alpha < p < 2$ (hence $T^{1,2,m}(dtdy)$ boundedness), where $p_c^\alpha < 1$ is given by

$$\frac{n}{2m} < v(\alpha, q) = \mathrm{Re}\,\alpha + \frac{1-\beta}{2} + \frac{n}{m}\left(\frac{1}{q} - \frac{1}{2}\right) = \frac{n}{m}\left(\frac{1}{p_c^\alpha} - \frac{1}{2}\right).$$

Here we apply Lemma 3.5 of [AKMP12] with $\beta = 0$. More precisely, in general we do not assume $L^q - L^2$ off-diagonal decay for the differentiated family but on the semigroup, we apply Lemma 3.5 of [AKMP12], in fact first to $\int_0^{t/2} \left(\frac{s}{t}\right)^\alpha \frac{1}{t} e^{-\left(\frac{3t}{4}-s\right)\Lambda} F_s ds$, and then use the $T^{1,2,m}(dtdy)$ boundedness of $\left\{t\Lambda e^{-\frac{t}{4}\Lambda}\right\}_{t>0}$ via its $L^2 - L^2$ off-diagonal decay.

Clearly, $\mathbf{M}_{\Lambda,0}^{+,2} = \mathbf{M}_\Lambda^{+,2}$. Using Stein's analytic interpolation applied to the operator family $\mathbf{M}_{\Lambda,\alpha}^{+,2}(F)$, with allowable dependence $e^{|\mathrm{Im}\,\alpha|}$, gives the tent space $T^{p,2,m}(dtdy)$ boundedness of $\mathbf{M}_\Lambda^{+,2}$ (hence $\mathbf{M}_\Lambda^+$) for $p > \frac{2nq}{2n+mq}$.

More precisely, the calculations to get q_* are as follows

$$\frac{1}{q_*} = \frac{\theta_q}{1} + \frac{1-\theta_q}{q} \quad \text{where} \quad 0 = \theta_q\left(\frac{n}{q'm} - \frac{1}{2}\right) + (1-\theta_q)\left(-\frac{1}{2}\right).$$

That is, q_* is obtained when $\theta_q = \frac{q'm}{2n} \leq 1$.

3.3.4 Proof of Lemma 3.2.3 on (RHI)

First, the case $p = 2$ follows simply from the uniform L^2 boundedness of the semigroup $\{e^{-t\Lambda}\}_{t>0}$. Now let $(t,x) \in \mathbb{R}_+^{1+n}$. Using Minkowski's inequality, $L^q - L^2$ off-diagonal estimates for $\{e^{-t\Lambda}\}_{t>0}$, and Hölder's inequality give that

$$\left(\fint_{B(x,t^{1/m})} |\mathbf{G}_\Lambda^+(F)(t,y)|^2 dy \right)^{1/2}$$

$$
\begin{aligned}
&\leq \fint_{[t/4,t/2]} \left(\fint_{B(x,t^{1/m})} |e^{-(t-s)\Lambda} F_s(y)|^2 \, dy \right)^{1/2} ds \\
&\lesssim \sum_{j=0}^{\infty} 2^{-jmN} \fint_{[t/4,t/2]} \left(t^{-\frac{n}{m}} \int_{2^j B(x,t^{1/m})} |F(s,y)|^q \, dy \right)^{1/q} ds \\
&\leq \sum_{j=0}^{\infty} 2^{-j(mN-\frac{n}{q})} \fint_{[t/4,t/2]} \left(\fint_{2^j B(x,t^{1/m})} |F(s,y)|^q \, dy \right)^{1/q} ds \\
&\leq \sum_{j=0}^{\infty} 2^{-j(mN-\frac{n}{q})} \left(\fint_{[t/4,t/2]} \left(\fint_{2^j B(x,t^{1/m})} |F(s,y)|^q \, dy \right)^{2/q} ds \right)^{1/2} \\
&\leq \left(\fint_{[t/4,t/2]} \left| M_q[F_s](x) \right|^2 ds \right)^{1/2},
\end{aligned}
$$

once we select $N > \frac{n}{mq}$. Taking the vertical square integrals $\left(\int_0^\infty |\cdot|^2 dt \right)^{1/2}$, then one can remove the average on $[t/4, t/2]$. Using Fubini's theorem, we thus have

$$
\left(\int_0^\infty \left(\fint_{B(x,t^{1/m})} |G(t,y)|^2 \, dy \right) dt \right)^{1/2} \lesssim \left(\int_0^\infty \left| M_q[F_s](x) \right|^2 ds \right)^{1/2}.
$$

Here $\forall\, x \in \mathbb{R}^n$

$$
M_q(\cdot)(x) := \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |\cdot|^q \right)^{1/q}.
$$

Hence, as $q < p$, and using Fefferman-Stein maximal theorem for the boundedness of M_q in $L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right)$, finally yield the desired conclusion.

3.4 Generalized Gaussian estimates

Following Blunck and Kunstmann, a class of uniformly $L^2(\mathbb{R}^n)$ bounded operators $\{T(z)\}_{z \in S_\delta}$, $0 < \delta < \pi/2$, is said to satisfy the generalized Gaussian-(q,r) estimates, with $1 \leq q \leq r \leq \infty$ and with homogeneity $m > 1$, if there exists $b > 0$ such that

$$
\left\| \mathbf{1}_{B\left(x,|z|^{\frac{1}{m}}\right)} T(z) \mathbf{1}_{B\left(y,|z|^{\frac{1}{m}}\right)} f \right\|_r \lesssim |z|^{-\frac{n}{m}\left(\frac{1}{q}-\frac{1}{r}\right)} e^{-b\left(\frac{|x-y|^m}{\operatorname{Re} z}\right)^{\frac{1}{m-1}}} \left\| \mathbf{1}_{B\left(y,|z|^{\frac{1}{m}}\right)} f \right\|_q
$$

for all $x, y \in \mathbb{R}^n$, all $z \in S_\delta$ and all $f \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Here we comment on the generalized Gaussian estimates for higher order elliptic operators of divergence or non-divergence form. The condition $m > 1$ in the above definition will not be a problem since we usually work with the homogeneity $m \in 2\mathbb{N}_+$.

Remark 3.4.1. As shown in [BK03], the generalized Gaussian-(q,r) estimates with homogeneity 2 and with certain $1 \le q \le r \le \infty$ also hold for the complex analytic semigroup $\{e^{-zL}\}_{z\in S_\delta}$, $0<\delta<\pi/2$, when L is a Schrödinger operator with a singular potential or an elliptic second order operator with singular lower order terms. We do not get into these issues here.

3.4.1 Divergence form elliptic operators

The materials of this subsection is based on Section 7.2 of [Aus07]. Consider an homogeneous elliptic operator L of order m, $m \in \mathbb{N}_+$, defined by

$$Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta}\partial^\beta f), \tag{3.4.1}$$

where the coefficients $a_{\alpha\beta}$ are complex-valued L^∞ functions on $\mathbb{R}^n$, and we assume

$$\left| \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x)\partial^\beta f(x)\partial^\alpha \bar{g}(x)dx \right| \le \widetilde{\lambda} \left\|\nabla^m f\right\|_2 \left\|\nabla^m g\right\|_2$$

and the strong Gårding inequality

$$\mathrm{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x)\partial^\beta f(x)\partial^\alpha \bar{f}(x)dx \ge \lambda \left\|\nabla^m f\right\|_2^2$$

for some $\lambda > 0$ and $\widetilde{\lambda} < \infty$ independent of $f, g \in W^{m,2}(\mathbb{R}^n)$. Here the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\partial_j = -i\partial/\partial x_j$ for $j = 1, \cdots, n$, and ∇^k is the array of all k-th order derivatives.

There exists an interval $\big(p_-(L), p_+(L)\big) \subset (1,\infty)$ for which the L^p boundedness of the semigroup $\{e^{-tL}\}_{t>0}$ holds once $p \in \big(p_-(L), p_+(L)\big)$. According to [Aus07, Proposition 3.2 and the remark that follows], for any $p_- < q \le r < p_+$ the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the $L^q - L^r$ off-diagonal estimates with homogeneity $2m$. The analytic extension results (from $L^q - L^r$ off-diagonal estimates for real time to generalized Gaussian-(q,r) estimates for complex time), with homogeneity $2m$, follow by arguments similar to the proof of [Aus07, Proposition 3.15]. We point out that $2 \in \big(p_-(L), p_+(L)\big)$.

3.4.2 Non-divergence form elliptic operators

The materials of this subsection is based on Section 3 of [Kun08]. These non-divergence elliptic operators of order $2m$ are of the form:

$$L = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha, \quad \mathbf{D}_q(L) = W^{2m,q}\left(\mathbb{R}^n\right), \quad 1<q<\infty, \tag{3.4.2}$$

where a_α are bounded measurable and complex-valued functions. Assume

$$\begin{aligned}(H)_q \quad & -L \text{ with } \mathbf{D}_q(L) = W^{2m,q} \\ & \text{is the generator of an analytic semigroup } T_q(\cdot) \text{ in } L^q.\end{aligned} \tag{3.4.3}$$

Here $\mathbf{D}_q(L)$ is the domain of the $L^q(\mathbb{R}^n)$ realization of the operator L.

The generalized Gaussian $(q,2)$-estimates for $T_q(\cdot)$ are satisfied when both $(H)_q$ and $(H)_2$ hold. See [Kun08, Theorem 1.1, Proposition 4.1 a) and b)].

3.4.3 Proof of Theorem 3.1.8

Let p_- be the infimum of q for which $-L$ generates a bounded analytic semigroup on $L^q(\mathbb{R}^n)$. According to what we commented in last two subsections, for $p_- < q < 2$, the semigroup $\{e^{-zL}\}_{z\in S_\delta}$, $\delta \in (0,\pi/2)$, for L an elliptic divergence form operator of order m or an elliptic non-divergence form operator of order $2m$, satisfy in particular the generalized Gaussian-$(q,2)$ estimates with order $2m$. The conclusion of Theorem 3.1.8 follows once we invoke part II) of Theorem 3.1.7, plus Theorem 3.1.6.

3.5 Extrapolation of R-analyticity

In the first subsection we give the proof of Theorem 3.1.9 on improved criteria for the extrapolation of R-boundedness. In the subsequent subsection we relate this extrapolation argument, in the particular case the extrapolation of R-analyticity, to the method used by S. Blunck and P. C. Kunstmann in [BK02]. In this regard, see also another method by F. Bernicot and J. Zhao in [BZ09].

3.5.1 Proof of Theorem 3.1.9 on R-boundedness

Under the same arguments as in the proof of Lemma 3.2.3, we can show

Lemma 3.5.1. *Let $1 \le q < 2$. Assume that the uniformly $L^2(\mathbb{R}^n)$ bounded operator family $\mathscr{T} = \{T(t)\}_{t>0}$ satisfies the $L^q - L^2$ off-diagonal decay with homogeneity m and with decay order $M^{\mathscr{T}} > \frac{n}{mq}$. Then*

$$\left\|(T(t)F_t)(y)\right\|_{T^{p,2,m}(dt\,dy)} \lesssim \|F\|_{L^p(\mathbb{R}^n;L^2(\mathbb{R}_+))}$$

for any $q < p \le 2$.

Then R-boundedness of $\{T(t)\}_{t>0}$ in $L^p(\mathbb{R}^n)$, $q < p \le 2$, is an easy consequence of the above lemma and the embedding $T^{p,2,m}(dt\,dy) \hookrightarrow L^p\left(\mathbb{R}^n; L^2(\mathbb{R}_+)\right)$ from [AHM12].

3.5.2 Relation with Blunck-Kunstmann criteria

First, we state the characterization of R-analyticity given in [KW04b, Theorem 2.20]:

Let $-\Lambda$ be the generator of a bounded analytic semigroup in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $\{e^{-z\Lambda}\}_{z\in S_\delta}$ for some $0 < \delta < \pi/2$ is R-bounded in $L^p(\mathbb{R}^n)$, is equivalent to, that

$\{e^{-t\Lambda}, t\Lambda e^{-t\Lambda}\}_{t>0}$ *is R-bounded in* $L^p(\mathbb{R}^n)$.

The latter condition is used as our definition of R-analyticity in the Introduction.

Now, the approach of Blunck-Kunstmann [BK02] to the R-analyticity in $L^p(\mathbb{R}^n)$, $q<p<2$, is to first prove some $L^p\left(\mathbb{R}^n; L^{2+\varepsilon}(\mathbb{R}_+)\right)$ boundedness of the real semigroup $\{e^{-t\Lambda}\}_{t>0}$, then uses the analyticity in $L^p(\mathbb{R}^n)$, which is the $L^p(\mathbb{R}^n; L^p(\mathbb{R}_+))$ boundedness of the complex semigroup $\{e^{-z\Lambda}\}_{z\in S_\delta}$ for some $0<\delta<\pi/2$, and finally interpolate to conclude for the R-boundedness of $\{e^{-z\Lambda}\}_{z\in S_{\widetilde{\delta}}}$ for some $0<\widetilde{\delta}<\delta$. The tool used in [BK02] is the $L^q - L^{2+\varepsilon}$ off-diagonal decay.

Then, the approach of [BK02] is improved by Kunstmann in [Kun08] using only the $L^q - L^2$ off-diagonal decay. But the decay order in [Kun08, Proposition 2.3] is still larger than the one required by us. More precisely, they use a threshold condition for the order of $L^q - L^2$ off-diagonal decay of the semigroup $\{e^{-z\Lambda}\}_{z\in S_\delta}$, in the case of homogeneity $m=1$, that $M>\frac{n}{q}+\frac{1}{q'}$. This improves the one $M>\frac{n}{q}+1$ used in [BK02]. Recall that in the case $m=1$ the threshold condition in our criteria is $M>\frac{n}{q}$.

Finally we remark that, although we only work with Gaussian estimates in this chapter, for an abstract semigroup the improvement on lowering the threshold required in the Blunck-Kunstmann criteria is still of independent interest.

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Part III

Holomorphic Operational Calculus —— Analysis of/on Whitney Averages

4

Weighted tent spaces with Whitney averages: strong factorization, complex interpolation and duality

Abstract

In this chapter, we introduce a new scale of tent spaces which covers, the tent spaces of Coifman-Meyer-Stein and of Hofmann-Mayboroda-McIntosh, and some other tent spaces considered by Dahlberg, Kenig-Pipher and Auscher-Axelsson for rough elliptic systems. The strong factorizations within our tent spaces, with applications to quasi-Banach complex interpolation and to multiplier-duality theory, are established. This way, we unify and extend the corresponding results obtained by Coifman-Meyer-Stein, Cohn-Verbitsky and Hytönen-Rosén.

Contents

4.1 Basic notation and chapter structure

Let $\mathbb{R}^{1+n}_+ = \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^n \times (0,\infty)$ be the usual upper half-space in $\mathbb{R}^{n+1}$. Points in $\mathbb{R}^n$ (respectively in $\mathbb{R}^{1+n}_+$) will be generally denoted by the letters x or z (respectively by (y,t) or (z,s)). For a point (y,t) in $\mathbb{R}^{1+n}_+$, we let $B(y,t) = \{z \in \mathbb{R}^n \mid |z-y| < t\}$ lie in the boundary $\mathbb{R}^n = \partial\mathbb{R}^{1+n}_+$. Here and below, the capital letter B denotes an open ball in $\mathbb{R}^n$, and $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^n$.

Given $\alpha > 0$, we shall denote the cone with aperture α and vertex $x \in \mathbb{R}^n$ by

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}^{1+n}_+ \mid |y-x| < \alpha t\} = \{(y,t) \in \mathbb{R}^{1+n}_+ \mid B(y,\alpha t) \ni x\},$$

and shall denote the tent with aperture α and base $B \subset \mathbb{R}^n$ by

$$\widehat{B^\alpha} := \left(\bigcup_{x\in B^c} \Gamma_\alpha(x)\right)^c = \{(y,t) \in \mathbb{R}^{1+n}_+ \mid B(y,\alpha t) \subset B\}.$$

If $\alpha = 1$, we simply write $\Gamma(x)$ and $\widehat{B}$.

Given a point $(y,t) \in \mathbb{R}^{1+n}_+$, we construct its *Whitney box* as

$$W(y,t) := \{(z,s) \in \mathbb{R}^{1+n}_+ \mid |z-y| < \alpha_1 t, \alpha_2^{-1} t < s < \alpha_2 t\}.$$

Here, the two numbers (α_1,α_2) with $\alpha_1 > 0$ and $\alpha_2 > 1$, are called the *Whitney parameters*. They are said to be *consistent* if $0 < \alpha_1 < \alpha_2^{-1} < 1$.

Throughout this chapter, the set of Vinogradov notations $\{\lesssim, \simeq, \gtrsim\}$ will be used. For two quantities a and b, which can be function values, set volumes, function norms or anything else, the term $a \lesssim b$ means that there exists a constant $C > 0$, which depends on parameters at hand, such that $a \le Cb$. In a similar way, $a \gtrsim b$ means $b \lesssim a$, and, $a \simeq b$ means both $a \lesssim b$ and $a \gtrsim b$.

This chapter is organized as follows.

— Section 4.2. We define in Definition 4.2.1 our scale of tent spaces $T^{p,r}_{q,\beta}$ systematically. At the end of this section we will also discuss some basic properties, such as convexity and separability, of these new tent spaces.

— Section 4.3. We show that the definition of our tent spaces is independent of the aperture used for cones and tents, and the pair of Whitney parameters used for Whitney boxes. As a reward, we can see for $r = q$ the coincidence (Theorem 4.3.2) of our tent spaces with the classical tent spaces of Coifman-Meyer-Stein and the weighted tent spaces of Hofmann-Mayboroda-McIntosh.

— Section 4.4 and Section 4.7. The central result of this chapter concerning an endpoint factorization theorem (Theorem 4.4.2) is presented in Section 4.4, with its full proof postponed to Section 4.7. Together with a multiplication lemma, we show the general multiplication and factorization theorem (Theorem 4.4.5) as a consequence of Theorem 4.4.2.

— Section 4.5 and Section 4.6. Under the general multiplication and factorization theorem, the quasi-Banach complex interpolation (Theorem 4.5.3) and the multiplier-duality results (Theorem 4.6.2 and Theorem 4.6.4) will be established in Section 4.5 and Section 4.6 respectively. There, we will also make a detailed connection with the corresponding known results on interpolation, multiplication, factorization and duality of tent spaces, which are mainly obtained by Coifman-Meyer-Stein, Cohn-Verbitsky and Hytönen-Rosén.

4.2 Definitions of the tent spaces $T^{p,r}_{q,\beta}$

Let $r \in (0,\infty]$. By $L^r_{\rm loc}(\mathbb{R}^{1+n}_+;\mathbb{C})$, we mean the class of complex-valued measurable functions which are defined on $\mathbb{R}^{1+n}_+$ and locally in L^r. This interpretation makes sense when $r=\infty$. Note that in general, we identify two measurable functions if they differ on a set with measure 0. For $r \in (0,\infty)$ and $f \in L^r_{\rm loc}(\mathbb{R}^{1+n}_+;\mathbb{C})$, denote the (unweighted) L^r*-Whitney average* of f on $W(y,t)$ by

$$\mathscr{W}_r(f)(y,t) := |W(y,t)|^{-1/r}\|f\|_{L^r(W(y,t),\,dzds)},$$

while for $r=\infty$, we take the usual essential supremum interpretation

$$\mathscr{W}_\infty(f)(y,t) := \operatorname*{ess\,sup}_{(z,s)\in W(y,t)} |f(z,s)|.$$

Here and below, apart from the Euclidean distance, $|\cdot|$ also denotes the moduli of complex values or the set volumes in $\mathbb{R}^n$ and $\mathbb{R}^{1+n}_+$.

Definition 4.2.1. *I*) For $0<p,q\le\infty$, we first define in $L^q_{\rm loc}(\mathbb{R}^{1+n}_+;\mathbb{C})$ the scale of *tent spaces* T^p_q according to the following four non-overlapping categories.

A) $0<p,q<\infty$. In this case, we let

$$T^p_q := \{g \mid \mathscr{A}_q(g)\in L^p(\mathbb{R}^n)\} \text{ and } \|g\|_{T^p_q} := \|\mathscr{A}_q(g)\|_{L^p},$$

where the *conical* q*-functional* $\mathscr{A}_q$ is defined as

$$\mathscr{A}_q(g)(x) := \left(\iint_{\Gamma(x)} |g(y,t)|^q \frac{dydt}{t^{n+1}}\right)^{1/q},\ x\in\mathbb{R}^n.$$

B) $0<q<p=\infty$. In this case, we let

$$T^\infty_q := \{g \mid \mathscr{C}_q(g)\in L^\infty(\mathbb{R}^n)\} \text{ and } \|g\|_{T^\infty_q} := \|\mathscr{C}_q(g)\|_{L^\infty},$$

where the *Carleson* q*-functional* $\mathscr{C}_q$ is defined as

$$\mathscr{C}_q(g)(x) := \sup_{B\ni x}|B|^{-1/q}\left(\iint_{\widehat{B}} |g(y,t)|^q \frac{dydt}{t}\right)^{1/q},\ x\in\mathbb{R}^n.$$

C) $0 < p < q = \infty$. In this case, we let

$$T^p_\infty := \{g \mid \mathcal{N}(g) \in L^p(\mathbb{R}^n)\} \text{ and } \|g\|_{T^p_\infty} := \|\mathcal{N}(g)\|_{L^p},$$

where the *non-tangential maximal functional* $\mathcal{N}$ is defined as

$$\mathcal{N}(g)(x) := \operatorname*{ess\,sup}_{(y,t)\in\Gamma(x)} |g(y,t)|,\ x \in \mathbb{R}^n.$$

D) $p = q = \infty$. In this case, we simply let $T^\infty_\infty := L^\infty(\mathbb{R}^{1+n}_+)$.

II) Let $\beta \in \mathbb{R}$. We also define the scale of *weighted tent spaces* $T^p_{q,\beta}$ by

$$T^p_{q,\beta} := \{g \mid g(y,t)t^{-\beta} \in T^p_q\} \text{ and } \|g\|_{T^p_{q,\beta}} := \|g(y,t)t^{-\beta}\|_{T^p_q}.$$

III) Given $0 < r \le \infty$ and $\beta \in \mathbb{R}$, and assume that the pair of Whitney parameters (α_1, α_2) is consistent. Then corresponding to each category above, we define in $L^r_{\mathrm{loc}}(\mathbb{R}^{1+n}_+; \mathbb{C})$ the scale of *tent spaces with Whitney averages* $T^{p,r}_q$ by

$$T^{p,r}_q := \{f \mid \mathcal{W}_r(f) \in T^p_q\} \text{ and } \|f\|_{T^{p,r}_q} := \|\mathcal{W}_r(f)\|_{T^p_q},$$

and the scale of *Weighted tent spaces with Whitney averages* $T^{p,r}_{q,\beta}$ by

$$T^{p,r}_{q,\beta} := \{f \mid f(z,s)s^{-\beta} \in T^{p,r}_q\} \text{ and } \|f\|_{T^{p,r}_{q,\beta}} := \|f(z,s)s^{-\beta}\|_{T^{p,r}_q}.$$

Convention 4.2.1'. Starting from Section 4.4 we modify the definition of $f \in T^p_\infty$ $(0 < p < \infty)$ via replacing "ess sup" by "sup" in $\mathcal{N}$ and assuming in addition f is everywhere defined. This change doesn't impact on continuous functions. For averaged tent spaces $T^{p,r}_\infty$ and $T^{p,r}_{\infty,\beta}$, this convention is consistent since the L^r-Whitney averages of L^r_{loc} functions are continuous. Also note that through the new definition of T^p_∞, we can not identify two functions if they agree almost everywhere.

In above definitions, the L^r-Whitney average and the weight β are required for the applications to boundary value problems of second order elliptic PDEs (see [AA11] for example). In practice β is a regularity index, and the weight constraint $\beta \in [-2/q, 0]$, with $\beta = 0$ if $q = \infty$, is taken in [AA11]. It is easy to see that $f \in T^p_\infty$ for $0 < p < \infty$ implies f is everywhere finite. Hence in the Category *C*), we identify two functions if they are equal everywhere. Moreover, in Type *III*) $\mathcal{W}_r(f)$ is also everywhere finite.

Remark 4.2.2. By definition $T^p_{q,0} = T^p_q$ and $T^{p,r}_{q,0} = T^{p,r}_q$. Moreover, $T^p_{q,\beta}$ is isometric to T^p_q and $T^{p,r}_{q,\beta}$ is isometric to $T^{p,r}_q$ via $f \to \tilde{f}$ with $\tilde{f}(z,s) = f(z,s)s^{-\beta}$. Also observe that since $(z,s) \in W(y,t)$ implies $s \simeq t$, we have that $f \in T^{p,r}_{q,\beta} \iff \mathcal{W}_r(f) \in T^p_{q,\beta}$.

Remark 4.2.3. 1) The classical tent spaces of Coifman-Meyer-Stein in [CMS85], where the weight $\beta = 0$ and Category *C*) is smaller, and the weighted tent spaces of Hofmann-Mayboroda-McIntosh in [HMM11], where only Category *A*) is considered, are all included in our scale $T^p_{q,\beta}$. More precisely, for Category *C*) [CMS85] uses "sup" instead of "ess sup" and requires the additional assumption $g \in C_{n.t.}(\mathbb{R}^{1+n}_+;\mathbb{C})$, meaning that g is a $\mathbb{C}$-valued continuous function on $\mathbb{R}^{1+n}_+$ and also has *non-tangential convergence*:

$$\lim_{\Gamma(x)\ni(y,t)\to x} g(y,t) \text{ exists for almost every } x \in \mathbb{R}^n.$$

ii) The scale $T^{p,r}_{q,\beta}$ with Whitney averages covers the function spaces which were introduced in [Dah86] and [KP93], and further investigated in [AA11], [HR13] and [Mou11]. In this regard, see the concluding paragraphs of Section 4.6 for a detailed correspondence. Note that we also bring in Category D). If $0 < r < \infty$, we call functions in $T^{\infty,r}_{\infty}$ the r-*Whitney multipliers*. In the trivial case $p = q = r = \infty$, it is not difficult to observe that $T^{\infty,\infty}_{\infty} = T^{\infty}_{\infty} = L^{\infty}(\mathbb{R}^{1+n}_+)$.

We end this section with several properties of our tent spaces.

Convexity and completeness. Given the tent space $T^{p,r}_{q,\beta}$, we let $\tau = \min(p,q,r)$. Observe that when $\tau \geq 1$, the space $T^{p,r}_{q,\beta}$ is Banach. In fact, the triangle inequality simply follows from Minkowski's inequality, and the completeness can be deduced from the one of T^p_q, as $\mathscr{W}_r(f(z,s)s^{-\beta}) \in T^p_q$ if $f \in T^{p,r}_{q,\beta}$.

Power and convexification. For a quasi-Banach function space, the trick of taking the powers is particularly useful. As for our tent spaces, let

$$[T^{p,r}_{q,\beta}]^{\theta} := \left\{ f \text{ measurable} \middle| |f|^{1/\theta} \in T^{p,r}_{q,\beta} \right\}, \theta \in (0,1),$$

equipped with $\|f\|_{[T^{p,r}_{q,\beta}]^{\theta}} := \left\| |f|^{1/\theta} \right\|^{\theta}_{T^{p,r}_{q,\beta}}$. This way, we have the realization

$$\left[T^{p,r}_{q,\beta} \right]^{\theta} = T^{p/\theta,r/\theta}_{q/\theta,\beta\theta}, \theta \in (0,1).$$

Now for the quasi-Banach $T^{p,r}_{q,\beta}$, with $\tau < 1$, $[T^{p,r}_{q,\beta}]^{\tau}$ is then a convexification of $T^{p,r}_{q,\beta}$.

Separability and density. Consider the covering of $\mathbb{R}^{1+n}_+$ by rational rectangles, which are of the product form $\prod_{i=1}^{n+1}(a_i,b_i)$, where for $1 \leq i \leq n+1$, a_i and b_i are in $\mathbb{Q}$, and $a_{n+1} > 0$. Let E be the linear span on $\mathbb{Q}$ of the characteristic functions of these rational rectangles. If $0 < p,q,r < \infty$, one can show that the countable set E is dense in $T^{p,r}_{q,\beta}$. We also point out that if $\sigma = \max(p,q,r) < \infty$, the L^r functions which have compact support in $\mathbb{R}^{1+n}_+$ are dense in $T^{p,r}_{q,\beta}$.

4.3 Coincidence and change of geometry

A demanding reader may ask two natural questions: i) how do the inner (local) Whitney averages $\mathscr{W}_r$ behave under the outer (boundary-reaching) $\mathscr{A}_q$ or $\mathscr{C}_q$ averages? ii) is our Definition 4.2.1 independent of the involved geometrical parameters? Aiming at the question i), we will first investigate the relation between the classical scale T^p_q

and our scale $T_q^{p,r}$ with Whitney averages. At the end of this section, we will also give an observation on the question ii).

Let us start with the following result.

Observation 4.3.1 (Change of apertures)**.** Define for $0<q<\infty$ and $\alpha>0$ the following three α-apertured functionals as

$$\mathscr{A}_q^\alpha(g)(x) := \left(\iint_{\Gamma_\alpha(x)} |g(y,t)|^q \frac{dydt}{t^{n+1}}\right)^{1/q}, x \in \mathbb{R}^n,$$

$$\mathscr{N}^\alpha(g)(x) := \operatorname*{ess\,sup}_{(y,t)\in\Gamma_\alpha(x)} |g(y,t)|, x \in \mathbb{R}^n,$$

$$\mathscr{C}_q^\alpha(g)(x) := \sup_{B\ni x} |B|^{-1/q} \left(\iint_{\widehat{B^\alpha}} |g(y,t)|^q \frac{dydt}{t}\right)^{1/q}, x \in \mathbb{R}^n.$$

Similar to Definition 4.2.1, these functionals can also result in a scale of tent spaces ${}^\alpha T_q^p$, where we let ${}^\alpha T_\infty^\infty = L^\infty$ for the trivial case $p=q=\infty$. It is well known that we have the change of aperture equivalence ${}^\alpha T_q^p = T_q^p$, with

$$C(n,\alpha,p,q)\|g\|_{T_q^p} \le \|g\|_{{}^\alpha T_q^p} \le C'(n,\alpha,p,q)\|g\|_{T_q^p}, 0<p,q\le\infty. \tag{4.3.1}$$

For the proof, see [FS72] for the simple situation $0<p<q=\infty$. For the case $q=2$ and $0<p<\infty$ (hence for $0<p,q<\infty$ by taking the powers of g properly), see [CMS85] for a rough, and [Tor86] for a refined argument on estimating C' when $\alpha>1$. By using atomic decomposition and interpolation, the sharp determination of both C and C' when $\alpha>0$, for the case $q=2$ and $0<p\le\infty$, has been recently obtained in [Aus11]. Note that the methods of [Aus11] extend to the case $q=\infty$ under minor modifications. We also remark that the vector-valued approach in [HTV91] and [HvNP08] can deal with the change of apertures in a very simple manner in the Banach case, and then a convexification process takes care of the quasi-Banach case, but this doesn't give optimal dependence in C'.

Theorem 4.3.2. *We have the coincidence with equivalence of quasi-norms*

$$T_{q,\beta}^{p,q} = T_{q,\beta}^p, 0<p,q\le\infty, \beta\in\mathbb{R}.$$

In particular, $T_q^{p,q} = T_q^p$, $0<p,q\le\infty$, showing that the classical tent spaces of Coifman-Meyer-Stein are included in the tent spaces with Whitney averages.

Proof. By Remark 4.2.2, it is enough to prove

$$T_q^{p,q} = T_q^p, 0<p,q\le\infty.$$

We start with the following *Whitney box geometry*: $\forall\,(z,s)\in\mathbb{R}_+^{1+n}$

$$W)\;\; W_*(z,s) \subset \{(y,t)|W(y,t)\ni(z,s)\} \subset W_{**}(z,s),$$

where W_* and W_{**} are the Whitney boxes associated to the Whitney parameters $(\alpha_1\alpha_2^{-1}, \alpha_2)$ and $(\alpha_1\alpha_2, \alpha_2)$ respectively[1], and (α_1, α_2) is the pair of Whitney parameters which defines W as in Definition 4.2.1. We only need to verify the choices of $\alpha_1\alpha_2^{-1}$ and $\alpha_1\alpha_2$, as the determination on α_2 is straightforward. To see the first inclusion in $W)$, given any $(y,t) \in W_*(z,s)$, we have $|z-y| < \alpha_1\alpha_2^{-1}s < \alpha_1 t$, which implies $W(y,t) \ni (z,s)$. To see the second inclusion, given any (y,t) with $W(y,t) \ni (z,s)$, we have $|y-z| < \alpha_1 t < \alpha_1\alpha_2 s$, which implies $(y,t) \in W_{**}(z,s)$. This proves the Whitney box geometry $W)$.

For the *cone geometry*, let $\alpha_0 = \alpha_2^{-1}(1-\alpha_1)$. We have that: $\forall\, x \in \mathbb{R}^n$

$$C_1) \quad (z,s) \in \Gamma_{\alpha_0}(x) \text{ and } (y,t) \in W_*(z,s) \Longrightarrow (y,t) \in \Gamma(x).$$

Indeed, we can compute as follow

$$|y-x| \le |y-z| + |z-x| < \alpha_1\alpha_2^{-1}s + \alpha_2^{-1}(1-\alpha_1)s < t.$$

Let $\alpha_C = \alpha_2 + \alpha_1\alpha_2$. There also holds: $\forall\, x \in \mathbb{R}^n$

$$C_2) \quad (y,t) \in \Gamma(x) \text{ and } (z,s) \in W(y,t) \Longrightarrow (z,s) \in \Gamma_{\alpha_C}(x).$$

Indeed, we can compute as follow

$$|z-x| \le |z-y| + |y-x| < \alpha_1 t + t < (\alpha_2 + \alpha_1\alpha_2)s.$$

Now from $W) + C_1)$, we have: $\forall\, x \in \mathbb{R}^n$

$$\chi_{\Gamma_{\alpha_0}(x)}(z,s)\chi_{W_*(z,s)}(y,t) \le \chi_{\Gamma(x)}(y,t)\chi_{W(y,t)}(z,s),$$

and from $W) + C_2)$, we have: $\forall\, x \in \mathbb{R}^n$

$$\chi_{\Gamma(x)}(y,t)\chi_{W(y,t)}(z,s) \le \chi_{\Gamma_{\alpha_C}(x)}(z,s)\chi_{W_{**}(z,s)}(y,t).$$

Then it follows from an integration in (y,t) that: $\forall\, x \in \mathbb{R}^n$

$$\chi_{\Gamma_{\alpha_0}(x)}(z,s) \lesssim \iint_{\mathbb{R}^{1+n}_+} \chi_{\Gamma(x)}(y,t)\frac{\chi_{W(y,t)}(z,s)}{t^{n+1}}\,dydt \lesssim \chi_{\Gamma_{\alpha_C}(x)}(z,s),$$

where in dividing by s^{n+1}, we use the similarity $s \simeq t$ implicitly.

If $0 < q < \infty$, multiplying by $|f(z,s)|^q$ the above inequalities and then integrating in (z,s), we have from Fubini's theorem that $\forall\, x \in \mathbb{R}^n$

$$\mathscr{A}_q^{\alpha_0}(f)(x) \lesssim \mathscr{A}_q(\mathscr{W}_q(f))(x) \lesssim \mathscr{A}_q^{\alpha_C}(f)(x).$$

If $q = \infty$, there holds a similar functional relation: $\forall\, x \in \mathbb{R}^n$

$$\mathscr{N}^{\alpha_0}(f)(x) \lesssim \mathscr{N}(\mathscr{W}_\infty(f))(x) \lesssim \mathscr{N}^{\alpha_C}(f)(x).$$

1. The pair of Whitney parameters defining W_{**} is not necessarily consistent, but for the purpose of geometric inclusions here, the consistency is not needed.

Note that for the left hand inequality we used implicitly the everywhere finiteness of f. For $0<p<\infty$, taking an L^p integration in x in the above two functional relations and using the change of aperture equivalence in Observation 4.3.1 lead us to the coincidence $T_q^{p,q}=T_q^p$ in Category A) and Category C).

For the *tent geometry*, let $\alpha_T=\alpha_2+\alpha_1\alpha_2^{-1}$. We have that: $\forall\, B\subset\mathbb{R}^n$

$$T_1)\ \ (z,s)\in\widehat{B^{\alpha_T}} \text{ and } (y,t)\in W_*(z,s)\Longrightarrow (y,t)\in\widehat{B}.$$

Indeed, given $B\subset\mathbb{R}^n$, $(z,s)\in\widehat{B^{\alpha_T}}$ and $(y,t)\in W_*(z,s)$, then $B(z,\alpha_T s)\subset B$. Thus

$$B(y,t)\subset B(z,t+|z-y|)\subset B(z,t+\alpha_1\alpha_2^{-1}s)\subset B(z,\alpha_T s),$$

so $B(y,t)\subset B$. Recall that $\alpha_0=\alpha_2^{-1}(1-\alpha_1)$. There also holds: $\forall\, B\subset\mathbb{R}^n$

$$T_2)\ \ (y,t)\in\widehat{B} \text{ and } (z,s)\in W(y,t)\Longrightarrow (z,s)\in\widehat{B^{\alpha_0}}.$$

Indeed, given $B\subset\mathbb{R}^n$, $(y,t)\in\widehat{B}$ and $(z,s)\in W(y,t)$, then $B(y,t)\subset B$. Thus

$$B(z,\alpha_0 s)\subset B(y,\alpha_0 s+|z-y|)\subset B(y,\alpha_0 s+\alpha_1 t)\subset B(y,t),$$

so $B(z,\alpha_0 s)\subset B$ and $(z,s)\in\widehat{B^{\alpha_0}}$.

Now from $W)+T_1)$, we have: $\forall\, B\subset\mathbb{R}^n$

$$\chi_{\widehat{B^{\alpha_T}}}(z,s)\chi_{W_*(z,s)}(y,t)\le\chi_{\widehat{B}}(y,t)\chi_{W(y,t)}(z,s),$$

and from $W)+T_2)$, we have: $\forall\, B\subset\mathbb{R}^n$

$$\chi_{\widehat{B}}(y,t)\chi_{W(y,t)}(z,s)\le\chi_{\widehat{B^{\alpha_0}}}(z,s)\chi_{W_{**}(z,s)}(y,t).$$

Then it follows from an integration in (y,t) that: $\forall\, B\subset\mathbb{R}^n$

$$\chi_{\widehat{B^{\alpha_T}}}(z,s)\lesssim\iint_{\mathbb{R}^{1+n}_+}\chi_{\widehat{B}}(y,t)\frac{\chi_{W(y,t)}(z,s)}{t^{n+1}}\,dy\,dt\lesssim\chi_{\widehat{B^{\alpha_0}}}(z,s),$$

where in dividing by s^{n+1}, we use again the similarity $s\simeq t$ implicitly.

If $0<q<\infty$, multiplying by $|f(z,s)|^q$ the above inequalities then integrating in (z,s) and taking a supremum over $B\ni x$, we have from Fubini's theorem that

$$\mathscr{C}_q^{\alpha_T}(f)(x)\lesssim\mathscr{C}_q(\mathscr{W}_q(f))(x)\lesssim\mathscr{C}_q^{\alpha_0}(f)(x),\ \forall\, x\in\mathbb{R}^n.$$

Taking an L^∞ norm in this functional relation and using the change of aperture in Observation 4.3.1 lead us to the coincidence $T_q^{p,q}=T_q^p$ in Category B).

Together with the trivial Category D), this concludes the proof. □

We end this section with another geometrical result, which will be needed in Section 4.7 for the proof of F_1) in Theorem 4.4.2.

Observation 4.3.3 (Change of Whitney parameters)**.** Note that we have frozen two consistent parameters (α_1, α_2) in Definition 4.2.1. Instead of considering different apertures as in Observation 4.3.1, here we replace (α_1, α_2) by another pair of consistent Whitney parameters (α_1', α_2'), with a prescribed chain condition

$$0 < \alpha_1 < \alpha_1' < 1/\alpha_2' < 1/\alpha_2 < 1.$$

Following Definition 4.2.1, we can also define a scale of tent spaces associated to (α_1', α_2'). Denoted by ${}^{(\alpha_1', \alpha_2')}T_q^{p,r}$, they should not be mistaken with ${}^{\alpha}T_q^p$ in Observation 4.3.1. We have the change of Whitney parameters equivalence

$$C(\alpha_1, \alpha_1', \alpha_2, \alpha_2')\|f\|_{T_q^{p,r}} \le \|f\|_{{}^{(\alpha_1', \alpha_2')}T_q^{p,r}} \le C'(\alpha_1, \alpha_1', \alpha_2, \alpha_2')\|f\|_{T_q^{p,r}}, \tag{4.3.2}$$

where the constants C and C' also implicitly depend on n, p, q and r.

The left hand inequality is straightforward from the chain condition satisfied by (α_1, α_2) and (α_1', α_2'). We prove the right hand inequality as follows. For $(y, t) \in \mathbb{R}_+^{1+n}$, denote $\widetilde{W}(y, t) = B(y, \gamma_1 t) \times (\gamma_2^{-1} t, \gamma_2 t)$, with $\gamma_1 \ge \alpha_1'/\alpha_1$ and $\gamma_2 \ge \alpha_2'/\alpha_2$. Then one can find an integer $N = N(n, \alpha_1, \alpha_2, \alpha_1', \alpha_2')$ such that, for any $(y, t) \in \mathbb{R}_+^{1+n}$, there exist a set of at most N points $\mathscr{P}_N(y, t)$ in $\widetilde{W}(y, t)$ with

$$\chi_{W'(y,t)}(z, s) \le \sum_{(\bar{y}, \bar{t}) \in \mathscr{P}_N(y,t)} \chi_{W(\bar{y}, \bar{t})}(z, s),$$

where W' is the Whitney box associated to the Whitney parameters (α_1', α_2'). Now using (4.3.1) in Observation 4.3.1 and the geometries $\{W), C_1), C_2), T_1), T_2)\}$ in proving Theorem 4.3.2, there exists $\alpha = \alpha(\alpha_1, \alpha_2, \alpha_1', \alpha_2')$ such that

$$\|f\|_{{}^{(\alpha_1', \alpha_2')}T_q^{p,r}} \lesssim \|f\|_{{}^{\alpha}T_q^{p,r}} \lesssim \|f\|_{T_q^{p,r}}.$$

We leave open the sharp determination on the bounds C and C' in (4.3.2).

4.4 Multiplication and factorization

Note that we are subjective to Convention 4.2.1' from now on.

The main goal of this chapter is to obtain, in the spirit of [CV00], the corresponding multiplication and factorization results for our new scale of tent spaces $T_{q,\beta}^{p,r}$. Some notations and definitions from function space theory are needed.

Denote by Σ the σ-finite measure space (Ω, μ), and by L^0 the collection of μ-measurable complex-valued functions on Ω. A *quasi-Banach function lattice X* on Σ is a non-empty subspace of L^0, which is equipped with a quasi-norm $\|\cdot\|_X$ such that, $(X, \|\cdot\|_X)$ is complete and X satisfies the *lattice property*: $\forall f \in X$,

$$\forall g \in L^0, \text{with } |g| \le |f| \ \ \mu-\text{a.e.}$$
$$\implies \ g \in X, \text{with } \|g\|_X \le \|f\|_X.$$

Clearly, for any f in a quasi-Banach function lattice X, $\|f\|_X = \||f|\|_X$.

Definition 4.4.1. Let $\{X_i\}_{i=0}^n$ be a collection of quasi-Banach function lattices.

M) By the multiplication: $X_0 \leftarrow X_1 \cdots X_n$, we mean that for any $f_i \in X_i$, $1 \le i \le n$, we have $f_1 \cdots f_n \in X_0$ and

$$\|f_1 \cdots f_n\|_{X_0} \lesssim \|f_1\|_{X_1} \cdots \|f_n\|_{X_n},$$

where the implicit constant is independent of $f_1, \cdots, f_n$.

F) By the (strong) factorization: $X_0 \rightarrow X_1 \cdots X_n$, we mean that for any $f_0 \in X_0$, there exist $f_i \in X_i$, $1 \le i \le n$, such that $|f_0| = |f_1| \cdots |f_n|$ and

$$\|f_1\|_{X_1} \cdots \|f_n\|_{X_n} \lesssim \|f_0\|_{X_0},$$

where the implicit constant is independent of $f_0, f_1, \cdots, f_n$.

When M) and F) are both satisfied, we write $X_0 \leftrightarrow X_1 \cdots X_n$.

In this chapter, our central task is to prove

Theorem 4.4.2. *For any $0 < p_0, q_0, r_0 \le \infty$, we have the following factorizations*

$$F_1)\ T_{q_0}^{p_0,r_0} \rightarrow T_{q_0}^{p_0,\infty} \cdot T_{\infty}^{\infty,r_0},$$

$$F_2)\ T_{q_0}^{p_0,r_0} \rightarrow T_{\infty}^{p_0,\infty} \cdot T_{q_0}^{\infty,r_0},$$

$$F_3)\ T_{q_0}^{p_0,r_0} \rightarrow T_{\infty}^{p_0,\infty} \cdot T_{q_0}^{\infty,\infty} \cdot T_{\infty}^{\infty,r_0}.$$

The proof of this endpoint factorization theorem will be postponed to Section 4.7. Meanwhile, there holds an endpoint multiplication result.

Lemma 4.4.3. *For any $0 < p_0, q_0, r_0 \le \infty$, we have the following multiplications*

$$M_1)\ T_{q_0}^{p_0} \leftarrow T_{\infty}^{p_0} \cdot T_{q_0}^{\infty},$$

$$M_2)\ T_{q_0}^{p_0,r_0} \leftarrow T_{\infty}^{p_0,\infty} \cdot T_{q_0}^{\infty,\infty} \cdot T_{\infty}^{\infty,r_0}.$$

Proof of Lemma 4.4.3. If $\max(p_0, q_0) = \infty$, there is nothing to prove for M_1). If $\max(p_0, q_0) < \infty$, then the multiplication M_1) is essentially in [CV00, Lemma 2.1]. The multiplication M_2) is a consequence of Hölder's inequality and M_1). In fact,

$$\begin{aligned} \|fgh\|_{T_{q_0}^{p_0,r_0}} &\le \|\mathcal{W}_\infty(f)\mathcal{W}_\infty(g)\mathcal{W}_{r_0}(h)\|_{T_{q_0}^{p_0}} \\ &\lesssim \|\mathcal{W}_\infty(f)\|_{T_\infty^{p_0}} \|\mathcal{W}_\infty(g)\|_{T_{q_0}^\infty} \|\mathcal{W}_{r_0}(h)\|_{T_\infty^\infty} \\ &= \|f\|_{T_\infty^{p_0,\infty}} \|g\|_{T_{q_0}^{\infty,\infty}} \|h\|_{T_\infty^{\infty,r_0}}, \end{aligned}$$

where f, g and h are all measurable functions on $\mathbb{R}_+^{1+n}$. □

Remark 4.4.4. Note that for M_1), the starting point of [CV00, Lemma 2.1] is the following inequality for Carleson measures (see [Ste93, p. 58–61] for example)

$$\iint_{\mathbb{R}_+^{1+n}} |f(y,t)|^p |d\mu|(y,t) \lesssim \|f\|_{T_\infty^p}^p \sup_{B \subset \mathbb{R}^n} \frac{|\mu|(\widehat{B})}{|B|}.$$

Note that the definition on T_∞^p here is the one formulated in the Convention 4.2.1'.

For $0 < p_1, p_2 \leq \infty$, define the *Hölderian relation* $(p_1, p_2)_H^{-1} = p_1^{-1} + p_2^{-1}$, where as usual, we will admit $1/\infty = 0$. Combining F_3) in Theorem 4.4.2 and M_2) in Lemma 4.4.3, we can deduce an intermediate claim where the Hölderian relations enter.

Theorem 4.4.5. *Suppose for $i \in \{0,1,2\}$, $T^{p_i,r_i}_{q_i,\beta_i}$ lies in the scale of weighted tent spaces with Whitney averages in Definition 4.2.1 and Convention 4.2.1'. Assume (H):*

$$p_0 = (p_1, p_2)_H,\ q_0 = (q_1, q_2)_H,\ r_0 = (r_1, r_2)_H \text{ and } \beta_0 = \beta_1 + \beta_2.$$

Then we have the multiplication and factorization

$$T^{p_0,r_0}_{q_0,\beta_0} \leftrightarrow T^{p_1,r_1}_{q_1,\beta_1} \cdot T^{p_2,r_2}_{q_2,\beta_2}.$$

Proof of Theorem 4.4.5. By Remark 4.2.2 and Definition 4.4.1, it is enough to assume $\beta_i = 0$, $i \in \{0,1,2\}$. Thus, we are only meant to show

$$T^{p_0,r_0}_{q_0} \leftrightarrow T^{p_1,r_1}_{q_1} \cdot T^{p_2,r_2}_{q_2}.$$

Call *extremal tent spaces* those $T^{p,r}_q$ with at least two among p, q, r equal to ∞. Therefore, $T^{p_0,r_0}_{q_0} \leftrightarrow T^{p_1,r_1}_{q_1} \cdot T^{p_2,r_2}_{q_2}$ holds trivially if $T^{p_0,r_0}_{q_0}$ is an extremal tent space. Indeed, multiplication is just a consequence of Hölder's inequality, and factorization follows from the trick of taking powers: $|f| = |f|^{1-\theta}|f|^{\theta}$, with $0 \leq \theta \leq 1$.

Now the general factorization can be proved as follows. With the Hölderian relation (H) in mind, factorizing $T^{p_0,r_0}_{q_0}$ through F_3) in Theorem 4.4.2 into extremal tent spaces, using the known factorization for extremal tent spaces, and multiplying through M_2) in Lemma 4.4.3, we then have

$$\begin{aligned} T^{p_0,r_0}_{q_0} &\to T^{p_0,\infty}_{\infty} \cdot T^{\infty,\infty}_{q_0} \cdot T^{\infty,r_0}_{\infty} \\ &\to T^{p_1,\infty}_{\infty} \cdot T^{p_2,\infty}_{\infty} \cdot T^{\infty,\infty}_{q_1} \cdot T^{\infty,\infty}_{q_2} \cdot T^{\infty,r_1}_{\infty} \cdot T^{\infty,r_2}_{\infty} \to T^{p_1,r_1}_{q_1} \cdot T^{p_2,r_2}_{q_2}. \end{aligned}$$

Finally, the general multiplication can be proved as follows. With the Hölderian relation (H) in mind, factorizing $T^{p_i,r_i}_{q_i}(i = 1,2)$ through F_3) in Theorem 4.4.2 into extremal tent spaces, using the known multiplication for extremal tent spaces, and multiplying through M_2) in Lemma 4.4.3, we then have

$$\begin{aligned} T^{p_1,r_1}_{q_1} \cdot T^{p_2,r_2}_{q_2} &\to T^{p_1,\infty}_{\infty} \cdot T^{\infty,\infty}_{q_1} \cdot T^{\infty,r_1}_{\infty} \cdot T^{p_2,\infty}_{\infty} \cdot T^{\infty,\infty}_{q_2} \cdot T^{\infty,r_2}_{\infty} \\ &\to T^{p_0,\infty}_{\infty} \cdot T^{\infty,\infty}_{q_0} \cdot T^{\infty,r_0}_{\infty} \to T^{p_0,r_0}_{q_0}. \end{aligned}$$

The quasi-norm inequalities in each proof can be obtained by inspection. □

4.5 quasi-Banach complex interpolation

We begin with a second look at the symbol "↔" for multiplication and factorization, which we formulated in last section in Definition 4.4.1.

Definition 4.5.1. Given two quasi-Banach function lattices X_1 and X_2, we define their *Calderón's product* $X_1 \bullet X_2$ as the class of $u \in L^0$ for which

$$\|u\|_{X_1 \bullet X_2} := \inf\{\|v\|_{X_1}\|w\|_{X_2} \mid |u| = |v||w|,\ v \in X_1,\ w \in X_2\} < \infty.$$

Clearly, the usual product $X_1 \cdot X_2 = \{vw \mid v \in X_1, w \in X_2\}$ is contained in the Calderón's product $X_1 \bullet X_2$, since $X_1 \bullet X_2$ is the completion of $X_1 \cdot X_2$ under the quasi-norm $\|\cdot\|_{X_1 \bullet X_2}$. Moreover, $X_0 \leftrightarrow X_1 \cdot X_2$ amounts to say $X_0 = X_1 \bullet X_2$, where we interpret the equality by the equivalence of quasi-norms.

This new product $X_1 \bullet X_2$, was first used by Calderón in [Cal64] as an intermediate space for the complex interpolation of a couple of Banach function lattices (X_1, X_2). For the underlying measure space $\Sigma = (\Omega, \mu)$, assume that Ω is a complete separable metric space, and μ is a σ-finite Borel measure on Ω. In a (most) natural extension of Calderón's interpolation method to the quasi-Banach setting, Kalton and Mitrea establish in [KM98, Section 3] (see also [Kal92]) that, for a couple of analytically convex separable quasi-Banach function lattices (X_1, X_2) on Σ, there holds the generalized *Calderón's product formula* (see [KM98, Theorem 3.4]) that

$$(X_1, X_2)_\theta = [X_1]^{1-\theta} \bullet [X_2]^\theta,\ 0 < \theta < 1.$$

Here, X analytically convex (A-convex for short) means that, for any analytic[2] function $\Phi : S = \{z \in \mathbb{C} \mid \operatorname{Re} z \in (0,1)\} \to X$, which is also continuous to the closed strip $\overline{S} = S \cup \partial S$, we have the *maximum modulus principle*

$$\max_{z \in S} \|\Phi(z)\|_X \lesssim \max_{z \in \partial S} \|\Phi(z)\|_X.$$

Under this A-convexity requirement, $X_1 + X_2$ is also A-convex, and then Calderón's method adapts to the quasi-Banach case. In the same spirit, this analytical approach to the interpolation of quasi-Banach function lattices was also considered in [BC91], where the ambient A-convex space is not necessarily the usual $X_1 + X_2$.

It was obtained in [Kal86] that X analytically convex is equivalent to X r-convex for some $r > 0$. Here, X *(lattice) r-convex* means that, for any $n \in \mathbb{N}_+$ and any $f_i \in X$, $i = 1, \dots, n$, we have the inequality

$$\left\| \left(\sum_{i=1}^n |f_i|^r \right)^{1/r} \right\|_X \le \left(\sum_{i=1}^n \|f_i\|_X^r \right)^{1/r}.$$

This convexification/normalization process is trivial for Banach function lattice X, as we can always take $r = 1$ in the above inequality. Thus for our purpose here, we can change A-convex to r-convex.

Now we turn to separability. A quasi-Banach function lattice X is said to satisfy the *Fatou property* [LT79, Remark 2 on p. 30], or *maximality in L^0*, if

$$\forall\, 0 \le f_n \in X \text{ and } \sup_{n \in \mathbb{N}_+} \|f_n\|_X < \infty, \text{ with } f_n \uparrow f \in L^0\ \mu-\text{a.e.}$$

2. See [KM98, p. 3911] for the precise definitions of analyticity and A-convexity.

$$\implies f \in X \text{ and } \|f\|_X = \lim_{n\to\infty} \|f_n\|_X.$$

It was observed in [Kal92] that, if both X_1 and X_2 satisfy the Fatou property, we only need to assume for the interpolation that either X_1 or X_2 is separable. In this regard, see also the second remark following of [KMM07, Theorem 7.9], where X_1 and X_2 are assumed to be sequence spaces. In fact, only the Fatou property of X_1 and X_2 is needed in the arguments there.

For further information on the applicability of Calderón's product formula, see [KM98, Section 3] and [KMM07, Section 7] directly. *Therefore, for two quasi-Banach function lattices X_1 and X_2, if X_i $(i = 1,2)$ is r_i-convex and has the Fatou property, and if either X_1 or X_2 is separable, then we have the desired interpolation realization:*

$$(X_1, X_2)_\theta = [X_1]^{1-\theta} \bullet [X_2]^\theta, \, 0 < \theta < 1.$$

Let us apply these to tent spaces.

Lemma 4.5.2. *All the tent spaces $T^{p,r}_{q,\beta}$ have the Fatou property.*

Proof. This is an easy consequence of the monotone convergence theorem and simple measure theoretic arguments. □

For $0 < p_1, p_2 \le \infty$ and $\theta \in (0,1)$, define the θ-*Hölderian relation* by $(p_1, p_2)_\theta^{-1} = (1-\theta)/p_1 + \theta/p_2$, where we again admit $1/\infty = 0$.

Theorem 4.5.3. *Let $0 < \theta < 1$. Suppose for $i \in \{0,1,2\}$, $T^{p_i,r_i}_{q_i,\beta_i}$ lies in the scale of weighted tent spaces with Whitney averages in Definition 4.2.1 and Convention 4.2.1'. Suppose* $\min\{\max\{p_1,q_1,r_1\},\max\{p_2,q_2,r_2\}\} < \infty$ *and assume the θ-Hölderian relation $(H)_\theta$:*

$$p_0 = (p_1, p_2)_\theta,\ q_0 = (q_1, q_2)_\theta,\ r_0 = (r_1, r_2)_\theta \text{ and } \beta_0 = (1-\theta)\beta_1 + \theta\beta_2.$$

Then under the Kalton-Mitrea complex interpolation method, we have

$$(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_2,r_2}_{q_2,\beta_2})_\theta = T^{p_0,r_0}_{q_0,\beta_0}.$$

Proof. With $(H)_\theta$ and Theorem 4.4.5, we have

$$T^{p_0,r_0}_{q_0,\beta_0} \leftrightarrow T^{p_1/(1-\theta),r_1/(1-\theta)}_{q_1/(1-\theta),\beta_1(1-\theta)} \cdot T^{p_2/\theta,r_2/\theta}_{q_2/\theta,\beta_2\theta},$$

which is equivalent to say

$$T^{p_0,r_0}_{q_0,\beta_0} = T^{p_1/(1-\theta),r_1/(1-\theta)}_{q_1/(1-\theta),\beta_1(1-\theta)} \bullet T^{p_2/\theta,r_2/\theta}_{q_2/\theta,\beta_2\theta}.$$

Under the condition $\min\{\max\{p_1,q_1,r_1\},\max\{p_2,q_2,r_2\}\} < \infty$, at least one quasi-Banach function lattice in the interpolation couple $(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_2,r_2}_{q_2,\beta_2})$ is separable. And it follows from Minkowski's inequality that, for $i = 1,2$, the quasi-Banach function lattice $T^{p_i,r_i}_{q_i,\beta_i}$ is $\min(\tau_i,1)$-convex, where $\tau_i = \min(p_i,q_i,r_i)$. In fact, it suffices to apply

$$\|f\|^{\tau_i}_{T^{p_i,r_i}_{q_i,\beta_i}} = \||f|^{\tau_i}\|_{T^{p_i/\tau_i,r_i/\tau_i}_{q_i/\tau_i,\beta_i\tau_i}},\ i = 1,2,$$

to the criterion of r-convexity, and notice that $T^{p_i/\tau_i, r_i/\tau_i}_{q_i/\tau_i,\beta_i\tau_i}$ $(i = 1,2)$ are Banach function lattices. Using the generalized Calderón's product formula, we have

$$\begin{aligned}(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_2,r_2}_{q_2,\beta_2})_\theta &= [T^{p_1,r_1}_{q_1,\beta_1}]^{1-\theta} \bullet [T^{p_2,r_2}_{q_2,\beta_2}]^\theta \\ &= T^{p_1/(1-\theta),r_1/(1-\theta)}_{q_1/(1-\theta),\beta_1(1-\theta)} \bullet T^{p_2/\theta,r_2/\theta}_{q_2/\theta,\beta_2\theta} = T^{p_0,r_0}_{q_0,\beta_0}.\end{aligned}$$

This proves the wanted complex interpolation formula. □

The above interpolation result is plausibly new, in view of the novel Whitney averaging factor. For the tent spaces without Whitney averages and with $\beta = 0$, the quasi-Banach complex interpolation

$$(T^{p_1}_{q_1}, T^{p_2}_{q_2})_\theta = T^{p_0}_{q_0}, \, 0 < \theta < 1,$$

where $1/p_0 = (1-\theta)/p_1 + \theta/p_2$ and $1/q_0 = (1-\theta)/q_1 + \theta/q_2$, was considered in [Ber92, Bernal], by another analytical method and for the almost full range $0 < p_1, p_2, q_1, q_2 < \infty$. For earlier results on the Banach complex interpolation, see the references in [Ber92]. Using the Kalton-Mitrea complex interpolation method, [CV00, Cohn-Verbitsky] recover the result in [Ber92]. For the weighted analogue of [CV00], see [HMM11, Hofmann-Mayboroda-McIntosh], where the weight $\beta \in \mathbb{R}$.

Here, by bringing in the endpoint space T^∞_∞, we have under Theorem 4.5.3 and the coincidence theorem that, for the full range $0 < p_1, p_2, q_1, q_2 \le \infty$, we have

$$(T^{p_1}_{q_1}, T^{p_2}_{q_2})_\theta = T^{p_0}_{q_0}, \, 0 < \theta < 1,$$

when $\min\{\max\{p_1,q_1,r_1\}, \max\{p_2,q_2,r_2\}\} < \infty$, $1/p_0 = (1-\theta)/p_1 + \theta/p_2$ and $1/q_0 = (1-\theta)/q_1 + \theta/q_2$. With this mild requirement $\min\{\max\{p_1,q_1,r_1\}, \max\{p_2,q_2,r_2\}\} < \infty$, we then cover all the complex interpolation results obtained in [Ber92] and [HMM11].

A last word on the separability condition for the interpolation pairs. For the case $\min\{\max\{p_1,q_1,r_1\}, \max\{p_2,q_2,r_2\}\} = \infty$, there exist some results in a different context. For $\alpha \in [0,1]$ and the space of Carleson measures of order α

$$V^\alpha := \left\{ d\mu \,\middle|\, \sup_{B\subset\mathbb{R}^n} \frac{|\mu|(\widehat{B})}{|B|^\alpha} < \infty \right\},$$

the complex interpolation $(V^0, V^1)_\alpha$ was identified in [AB79, Theorem 3-(ii)] to a space which is strictly smaller than V^α. See also [AM87] and [AM88] for relevant results.

4.6 Multipliers and standard duality

Now we turn to the *multiplier* issue, which from the *multiplication* point of view, is more straightforward than the quasi-Banach complex interpolation.

Similarly to last section, we restrict ourselves to the setting of (Banach) function lattices, and the underlying measure space $\Sigma = (\Omega, \mu)$ is assumed to be complete and σ-finite. Here, "complete" is with respect to the measure, meaning that

$$\forall\, E \subset \Omega, \mu(E) = 0 \Longrightarrow \forall\, E' \subset E, \mu(E') = 0.$$

Recall that L^0 is the collection of all complex-valued μ-measurable functions on Ω.

Definition 4.6.1. Given two Banach function lattices X_0 and X_1, we say that $w \in L^0$ is a *multiplier* from X_1 to X_0, if the associated multiplication mapping

$$M_w : X_1 \to X_0, v \mapsto vw$$

satisfies

$$\|M_w\|_{X_1 \to X_0} := \sup_{v \neq 0} \frac{\|vw\|_{X_0}}{\|v\|_{X_1}} < \infty.$$

Denote all the multipliers from X_1 to X_0 by $M(X_1, X_0)$, equipped with

$$\|w\|_{M(X_1,X_0)} = \|M_w\|_{X_1 \to X_0}.$$

Before proceeding to our main results in this section, we review a cancellation result concerning Calderón's product. It was obtained in [Sch10, Theorem 2.5 and Corollary 2.6] that for three Banach function lattices $\{E, F, G\}$ on Σ, all with the Fatou property, we have the following *cancellation formula*

$$E \bullet F = E \bullet G \Longrightarrow F = G.$$

There also holds (see [Sch10, Theorem 2.8]) that

$$F = M(E, E \bullet F),$$

if both E and F have the Fatou property. In particular situations, the above multiplier representation can also be found in [CNS03, Theorem 3.5], which served to prove the uniqueness theorem of Calderón-Lozanovskii's interpolation method. We mention that in the literature, the construction of Calderón for intermediate spaces was further investigated by Lozanovskii in a series of papers ([Loz69], [Loz72]).

Let us apply these to our tent spaces.

Theorem 4.6.2. *With the same assumptions as in Theorem 4.4.5 and* $1 \le p_i, q_i, r_i \le \infty$ *for* $i \in \{0, 1, 2\}$*, we have the multiplier identification*

$$T^{p_2,r_2}_{q_2,\beta_2} = M(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_0,r_0}_{q_0,\beta_0}).$$

Proof. For $i \in \{0, 1, 2\}$, $1 \le p_i, q_i, r_i \le \infty$ implies $\tau_i = \min(p_i, q_i, r_i) \ge 1$, thus $T^{p_i,r_i}_{q_i,\beta_i}$ is a Banach function lattice. Using the multiplier representation cited above, with the Fatou property guaranteed by Lemma 4.5.2, we have

$$T^{p_2,r_2}_{q_2,\beta_2} = M(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_1,r_1}_{q_1,\beta_1} \bullet T^{p_2,r_2}_{q_2,\beta_2}) = M(T^{p_1,r_1}_{q_1,\beta_1}, T^{p_0,r_0}_{q_0,\beta_0}),$$

where the last equality is from Theorem 4.4.5: $T^{p_0,r_0}_{q_0,\beta_0} = T^{p_1,r_1}_{q_1,\beta_1} \bullet T^{p_2,r_2}_{q_2,\beta_2}$. □

Finally, we look at the duality theory. Given $\beta_0 \in \mathbb{R}$, we will consider the following β_0-weighted pairing

$$(f, h)_{\beta_0} := \iint_{\mathbb{R}^{1+n}_+} f(y,t) h(y,t) t^{-\beta_0 - 1} \, dy dt.$$

Let p', q' and r' be the dual indice of $1 \le p, q, r \le \infty$.

Definition 4.6.3. The β_0-weighted *Köthe dual* of the Banach $T^{p,r}_{q,\beta}$ is defined as

$$(T^{p,r}_{q,\beta})^*_{\beta_0} := M(T^{p,r}_{q,\beta}, L^1(\mathbb{R}^{1+n}_+, t^{-\beta_0-1}dydt)) = M(T^{p,r}_{q,\beta}, T^{1,1}_{1,\beta_0}).$$

Here, unlike the continuous functional dual $(\cdot)'$, "Köthe" means the dual within the class of Banach function lattices. For a general account on this aspect, see [LT79]. By the *standard duality*, we mean the (Köthe) dual of the Banach $T^{p,r}_{q,\beta}$ when $1 \le p < \infty$, $\beta \in \mathbb{R}$ and particularly $1 \le \min(q,r) \le \max(q,r) < \infty$.

Theorem 4.6.4. *Under the pairing $(\cdot,\cdot)_{\beta_0}$, we have the following standard duality*

$$T^{p',r'}_{q',\beta_0-\beta} = (T^{p,r}_{q,\beta})', \, 1 \le p,q,r < \infty, \, \beta \in \mathbb{R}.$$

Proof. By Theorem 4.6.2 and the definition of $(\cdot)^*_{\beta_0}$, we have

$$T^{p',r'}_{q',\beta_0-\beta} = M(T^{p,r}_{q,\beta}, T^{1,1}_{1,\beta_0}) = (T^{p,r}_{q,\beta})^*_{\beta_0} \subset (T^{p,r}_{q,\beta})',$$

where the last inclusion follows from the straightforward identification of multipliers to continuous linear functionals, through the pairing $(\cdot,\cdot)_{\beta_0}$.

For the converse, suppose that we are given a continuous linear functional l on $T^{p,r}_{q,\beta}$. Then whenever K is a compact set in $\mathbb{R}^{1+n}_+$, and whenever f is supported in K, with $f \in L^r(K)$, then $\mathscr{W}_r(f) \in T^p_{q,\beta}$ with

$$\|f\|_{T^{p,r}_{q,\beta}} = \|\mathscr{W}_r(f)\|_{T^p_{q,\beta}} \le C_K \|f\|_{L^r}.$$

Here, C_K is a constant which depends on the compact set K, and also implicitly on the indice p, q, r and β. Thus l induces a continuous linear functional on $L^r(K)$ and is representable by $h^K \in L^{r'}(K)$, as $1 \le r < \infty$. Taking an increasing family of such K which exhausts $\mathbb{R}^{1+n}_+$, gives us an $h \in L^{r'}_{\mathrm{loc}}$ such that

$$l(f) = (f,h)_{\beta_0} = \iint_{\mathbb{R}^{1+n}_+} f(y,t)h(y,t)t^{-\beta_0-1}dydt,$$

whenever $f \in L^r$ and has compact support. By density arguments, this representation of l by h extends to all $f \in T^{p,r}_{q,\beta}$, as we further have $1 \le p,q < \infty$. By the representation through $(\cdot,h)_{\beta_0}$, we have $\|l\| = \|M_H\|_{T^{p,r}_{q,\beta} \to T^{1,1}_{1,\beta_0}}$, which means

$$(T^{p,r}_{q,\beta})' \subset M(T^{p,r}_{q,\beta}, T^{1,1}_{1,\beta_0}) = (T^{p,r}_{q,\beta})^*_{\beta_0} = T^{p',r'}_{q',\beta_0-\beta}.$$

This then proves the desired standard duality. □

To end this section, we deduce as corollaries some corresponding known results on multiplication, factorization and duality, mainly obtained in the articles [CMS85, Coifman-Meyer-Stein], [CV00, Cohn-Verbitsky] and [HR13, Hytönen-Rosén].

Relation with Coifman-Meyer-Stein. For the standard duality, it was shown in [CMS85, Theorem 1-(b) and Theorem 2] that

$$T_2^{p'} = (T_2^p)_0^* = (T_2^p)', \; 1 \le p < \infty,$$

which upon using Theorem 4.3.2 on the coincidence for $r = q = 2$, then corresponds to our Theorem 4.6.4 in the particular case

$$T_{2,0}^{p',2} = \left(T_{2,0}^{p,2}\right)_0^* = \left(T_{2,0}^{p,2}\right)', \; 1 \le p < \infty.$$

By the *Carleson duality*, we mean the continuous functional dual of $T_{q,\beta}^{p,r}$ for $1 \le p < \infty$, $\beta \in \mathbb{R}$ and particularly $1 \le \min(q,r) \le \max(q,r) = \infty$. Let $\underline{\widehat{B}} := \overline{\widehat{B}}$ be the closed tent on base B, and denote the Carleson measures on $\overline{\mathbb{R}_+^{1+n}}$ by

$$\overline{\mathscr{C}} := \left\{ d\mu \middle| \sup_{B \subset \mathbb{R}^n} \frac{|\mu|\left(\underline{\widehat{B}}\right)}{|B|} < \infty \right\}.$$

Let $\mathscr{N}_0 = T_\infty^1 \cap C_{n.t.}$ be the tent space of [CMS85]. The classical Carleson duality ([CMS85, Proposition 1], see also [Ste93, page 63]) states that

$$\overline{\mathscr{C}} = \left(\mathscr{N}_0\right)'.$$

Obviously, our Theorem 4.6.4 on standard duality can *not* cover the Carleson duality. Nevertheless, we shall mention in Remark 6.2 a consequence of our method of proof toward factorization of bounded Borel measures on $\overline{\mathbb{R}_+^{1+n}}$ by Carleson measures.

Relation with Hytönen-Rosén. To relate their notations, $N_{p,q}$ and $C_{p',q'}$ in [HR13] for Banach cases are just the scales $T_{\infty,0}^{p,q}$ and $T_{1,-1}^{p',q'}$ here, and their duality claim is

$$N_{p,q} = (C_{p',q'})', \; 1 < p < \infty, \; 1 < q \le \infty.$$

This *Carleson (pre-)duality*, stated in [HR13, Theorem 3.2], then corresponds to our Theorem 4.6.4 in the particular case

$$T_{\infty,0}^{p,r} = (T_{1,-1}^{p',r'})_{-1}^* = (T_{1,-1}^{p',r'})', \; 1 < p < \infty, \; 1 < r \le \infty.$$

At the multiplication side, Theorem 3.1 of [HR13] states

$$T_{r,-1/r}^{r,r} \leftarrow T_{\infty,0}^{p,q} \cdot T_{r,-1/r}^{\widetilde{p},\widetilde{q}}, \; 1 \le r < \infty, \; r \le p < \infty, \; r \le q \le \infty,$$

with $r = (p, \widetilde{p})_H = (q, \widetilde{q})_H$. Again, this is a particular case of our Theorem 4.4.5.

Relation with Cohn-Verbitsky. Under the coincidence theorem and Remark 4.7.3, part F_2) in Theorem 4.4.2 for $r_0 = q_0$ corresponds to Cohn-Verbitsky

$$T_{q_0}^{p_0} = T_{q_0}^{p_0,q_0} \to T_\infty^{p_0,\infty} \cdot T_{q_0}^{\infty,q_0} = T_\infty^{p_0} \cdot T_{q_0}^\infty.$$

Meanwhile, with the help of F_1) to produce Whitney multipliers, our result F_3) is a further (polarized) factorization of the tent space $T_{q_0}^{p_0,r_0}$. Of course, we also bring in the endpoint spaces T^∞_∞ and T^{∞,r_0}_∞, which makes the statement broader. Moreover, we continue with a multiplier discussion basing on the factorization result, which is seemingly new even in the situation of classical tent spaces.

We also remark that the multiplication side of Theorem 4.4.5 covers Lemma 5.5 in [AA11] and Lemma 2.4.3 in [Mou11]. To relate the notations again, the tent spaces concerning gradients of solutions and Carleson perturbations in [AA11], originally introduced by Kenig-Pipher in [KP93] and by Dahlberg in [Dah86] respectively, correspond to $T^{2,2}_{\infty,0}$ and $T^{\infty,\infty}_{2,0}$ here. Our full scale $T^{p,r}_{q,\beta}$, mainly interested by $T^{p,2}_{\infty,0}$ and $T^{p,2}_{2,-1}$ for p in some interval containing 2, will be used as *natural function spaces* in part of a continuation work of [AA11], where more backgrounds on boundary value problems of elliptic PDEs can be referred.

4.7 Proof of Theorem 4.4.2 on factorization

To prove F_3) it suffices to show F_1) and F_2) respectively. Indeed, factorizing $T^{p_0,r_0}_{q_0}$ through F_1) first, then using F_2) yields F_3) immediately. Thus to prove Theorem 4.4.2, we show F_1) and F_2) in order.

Proof of F_1). Let $W^*(y,t)$ and $\mathscr{W}_r^*(\cdot)(y,t)$ be the Whitney box and the L^r-Whitney average associated to the point $(y,t)\in\mathbb{R}^{1+n}_+$, and to the Whitney parameters

$$\alpha_1^* = \alpha_1\big(1+\alpha_2^{1/2}\big)^{-1} \text{ and } \alpha_2^* = \alpha_2^{1/2},$$

where (α_1,α_2) is the pair of consistent Whitney parameters we used in Definition 4.2.1. Similarly, let W^{**} and $\mathscr{W}_r^{**}(\cdot)$ be the Whitney objects associated to

$$\alpha_1^{**} = \alpha_1\Big[2\big(1+\alpha_2^{1/2}\big)\alpha_2^{1/4}\Big]^{-1} \text{ and } \alpha_2^{**} = \alpha_2^{1/4}.$$

Note that the two resulted pairs of Whitney parameters are also consistent, with

$$0<\alpha_1^{**}<\alpha_1^*<\alpha_1<\alpha_2^{-1}<(\alpha_2^*)^{-1}<(\alpha_2^{**})^{-1}<1.$$

Moreover, for any $(y,t)\in\mathbb{R}^{1+n}_+$, we have the geometrical relations

$$\bigcup_{(z,s)\in W^*(y,t)} W^*(z,s)\subset W(y,t) \tag{4.7.1}$$

and

$$\bigcap_{(z,s)\in W^{**}(y,t)} W^*(z,s)\supset W^{**}(y,t). \tag{4.7.2}$$

The verification on α_2^* and α_2^{**} is straightforward. For the first inclusion, given any $(z,s)\in W^*(y,t)$ and any $(z_0,s_0)\in W^*(z,s)$, we have

$$|z_0-y|\le|z_0-z|+|z-y|<\alpha_1^* s+\alpha_1^* t<\alpha_1^*(\alpha_2^*+1)t=\alpha_1 t,$$

which implies $(z_0, s_0) \in W(y,t)$. For the second inclusion, given any $(z_0, s_0) \in W^{**}(y,t)$ and any $(z,s) \in W^{**}(y,t)$, we have

$$|z_0 - z| \le |z_0 - y| + |y - z| < 2\alpha_1^{**} t < 2\alpha_1^{**}\alpha_2^{**} s = \alpha_1^* s,$$

which implies $(z_0, s_0) \in W^*(z,s)$. This proves the two relations (4.7.1) and (4.7.2).

Now for any $u \in T_{q_0}^{p_0,r_0}$, we construct $v = \mathcal{W}_{r_0}^*(u)$. Then we have from (4.7.1) that

$$\sup_{(z,s)\in W^*(y,t)} \mathcal{W}_{r_0}^*(u)(z,s) \lesssim \mathcal{W}_{r_0}(u)(y,t)$$

is valid for any $(y,t) \in \mathbb{R}_+^{1+n}$, thus we know

$$\mathcal{W}_\infty^*(v) \lesssim \mathcal{W}_{r_0}(u) \text{ and } \|\mathcal{W}_\infty^*(v)\|_{T_{q_0}^{p_0}} \lesssim \|u\|_{T_{q_0}^{p_0,r_0}}.$$

For $w = u/\mathcal{W}_{r_0}^*(u)$, we then have from (4.7.2) that

$$\inf_{(z,s)\in W^{**}(y,t)} \mathcal{W}_{r_0}^*(u)(z,s) \gtrsim \mathcal{W}_{r_0}^{**}(u)(y,t)$$

is valid for any $(y,t) \in \mathbb{R}_+^{1+n}$, thus we know

$$\mathcal{W}_{r_0}^{**}(w) \lesssim 1 \text{ and } \|\mathcal{W}_{r_0}^{**}(w)\|_{T_\infty^\infty} \lesssim 1.$$

Using the change of Whitney parameters equivalence in Observation 4.3.3, $u = vw$ is then the desired factorization for $T_{q_0}^{p_0,r_0} \to T_{q_0}^{p_0,\infty} \cdot T_\infty^{\infty,r_0}$, $0 < p_0, q_0, r_0 \le \infty$. □

Proof of F_2). Observe that we can suppose $0 < \max(p_0, q_0) < \infty$. In fact, nothing has to be done if $p_0 = \infty$, and the case $q_0 = \infty$ is already included in F_1).

We base our arguments on the constructive proof in [CV00]. From the consistency of Whitney parameters, we have $0 < \alpha_1 < \alpha_2^{-1} < 1$. Then the following relations

$$\bigcap_{(z,s)\in W(y,t)} B(z,s) \supset B(y, (\alpha_2^{-1} - \alpha_1)t) \tag{4.7.3}$$

and

$$\bigcup_{(z,s)\in W(y,t)} B(z,s) \subset B(y, (\alpha_2 + \alpha_1)t) \tag{4.7.4}$$

hold for any $(y,t) \in \mathbb{R}_+^{1+n}$. In fact, for the verification of the first inclusion, given any $x \in B(y, (\alpha_2^{-1} - \alpha_1)t)$ and any $(z,s) \in W(y,t)$, we compute as follow

$$|x - z| \le |x - y| + |y - z| < (\alpha_2^{-1} - \alpha_1)t + \alpha_1 t < s,$$

which implies $x \in B(z,s)$. Similarly, to verify the second inclusion, given any $(z,s) \in W(y,t)$ and any $x \in B(z,s)$, we compute as follow

$$|x - y| \le |x - z| + |z - y| < s + \alpha_1 t < (\alpha_2 + \alpha_1)t,$$

which implies $x \in B(y, (\alpha_2 + \alpha_1)t)$. This proves the two relations (4.7.3) and (4.7.4).

As $0 < \max(p_0, q_0) < \infty$, the tent space $T^{p_0,r_0}_{q_0}$ lies in Category A) and can be determined by the conical functional $\mathscr{A}_{q_0}$. Therefore, $\tilde{u} = \mathscr{A}_{q_0}(\mathscr{W}_{r_0}(u)) \in L^{p_0}(\mathbb{R}^n)$. Denote by $P_0[h](y,t)$ the average of h on $B(y,t) \subset \mathbb{R}^n$, and construct $v = P_0[\tilde{u}^{\tilde{p}}]^{1/\tilde{p}}$ for some $\tilde{p} < p_0$. Let $\alpha^* = \alpha_2 + \alpha_1 > 1$, then by (4.7.4), for any $(y,t) \in \mathbb{R}^{1+n}_+$

$$\sup_{(z,s)\in W(y,t)} v(z,s) \lesssim v(y,\alpha^* t) =: v^*(y,t).$$

Thus we have $\mathscr{W}_\infty(v)(y,t) \lesssim v^*(y,t)$, and there holds

$$\mathscr{N}(\mathscr{W}_\infty(v))(x) \lesssim \mathscr{N}(v^*)(x) \le \mathbf{M}(\tilde{u}^{\tilde{p}})^{1/\tilde{p}}(x),\ \forall\, x \in \mathbb{R}^n,$$

where $\mathscr{N}$ is the non-tangential maximal functional, $\mathbf{M}$ is the Hardy-Littlewood maximal operator and the last estimate follows from the fact

$$\bigcap_{(y,t)\in\Gamma(x)} B(y,\alpha^* t) \ni x,\ \forall\, x \in \mathbb{R}^n.$$

As $p_0/\tilde{p} > 1$, then by maximal theorem, we have

$$\|v\|_{T^{p_0,\infty}_\infty} \lesssim \|\mathbf{M}(\tilde{u}^{\tilde{p}})^{1/\tilde{p}}\|_{L^{p_0}} \lesssim \|\tilde{u}\|_{L^{p_0}} = \|u\|_{T^{p_0,r_0}_{q_0}}.$$

Now we turn to $w = u/v$. Let $\alpha_* = \alpha_2^{-1} - \alpha_1 \in (0,1)$, then by (4.7.3)

$$\inf_{(z,s)\in W(y,t)} v(z,s) \gtrsim v(y,\alpha_* t)$$

is valid for any $(y,t) \in \mathbb{R}^{1+n}_+$. By Hölder's inequality, there holds

$$\|h^{-1}\|^{-1}_{L^q(dv)} \le \|h\|_{L^r(dv)},\ \forall\, q > 0,\ \forall\, r > 0, \tag{4.7.5}$$

when dv is a probability measure on $\mathbb{R}^n$. Applying this estimate with $h = \tilde{u}$, $r = \tilde{p}$, $q = q_0$ and $dv(x) = |B(y,\alpha_* t)|^{-1}\chi_{B(y,\alpha_* t)}(x)dx$, we have for any $(y,t) \in \mathbb{R}^{1+n}_+$

$$\begin{aligned}\inf_{(z,s)\in W(y,t)} v(z,s) &\gtrsim P_0[\tilde{u}^{\tilde{p}}]^{1/\tilde{p}}(y,\alpha_* t)\\ &\ge P_0[\tilde{u}^{-q_0}]^{-1/q_0}(y,\alpha_* t) \gtrsim P_0[\tilde{u}^{-q_0}]^{-1/q_0}(y,t),\end{aligned}$$

where the last estimate follows from $0 < \alpha_* < 1$ and $-1/q_0 < 0$. We write $\|\cdot\|_c = \|\cdot\|_{T^\infty_{1,-1}}$ for the Carleson norm of measurable functions on $\mathbb{R}^{1+n}_+$, and let

$$d\mu(y,t) = \mu(y,t)\,dy\,dt = \mathscr{W}_{r_0}(u)^{q_0}(y,t)\,t^{-1}\,dy\,dt.$$

The above pointwise estimates on v further imply

$$\begin{aligned}\|\mathscr{W}_{r_0}(u/v)\|_{T^\infty_{q_0}} &\lesssim \|P_0[\tilde{u}^{-q_0}]^{1/q_0}\mathscr{W}_{r_0}(u)\|_{T^\infty_{q_0}}\\ &= \|P_0[\tilde{u}^{-q_0}]\mu\|^{1/q_0}_{T^\infty_{1,-1}} = \|P_0[\mathscr{A}_1(\mu(y,t)t)^{-1}]\mu\|^{1/q_0}_c \lesssim 1.\end{aligned}$$

In the last estimate, we used the lemma below.

Therefore, we can conclude the proof of F_2). □

We record down the missing part in estimating $\|P_0[\mathscr{A}_1(\mu(y,t)t)^{-1}]\mu\|_c \lesssim 1$. For a non-negative measure $d\mu$ on $\overline{\mathbb{R}^{1+n}_+}$, denote its (free) balayage by

$$\overline{\mathscr{A}}(d\mu)(x) := \iint_{\Gamma(x)} \frac{d\mu(z,s)}{s^n},\ x \in \mathbb{R}^n.$$

This way, we can reconstruct from the boundary value $\overline{\mathscr{A}}(d\mu)$ its (free) extension

$$E(d\mu)(y,t) := P_0[\overline{\mathscr{A}}(d\mu)^{-1}](y,t),\ \forall\,(y,t) \in \mathbb{R}^{1+n}_+.$$

Thus in the desired estimate, with $d\mu(y,t) = \mu(y,t)\,dy\,dt$ supported in $\mathbb{R}^{1+n}_+$, we have

$$P_0[\mathscr{A}_1(\mu(z,s)s)^{-1}](y,t)\mu(y,t)\,dy\,dt = E(d\mu)(y,t)\,d\mu(y,t).$$

The next lemma is very simple and can be found in [CV00, Lemma 2.2], or one can refer to [AB79] directly. For the completeness, we still provide an argument here. Recall that $\widehat{B}$ denotes the closed tent with base $B \subset \mathbb{R}^n$.

Lemma 4.7.1. *For any non-negative measure $d\mu$ on $\overline{\mathbb{R}^{1+n}_+}$, we have*

$$\|E(d\mu)d\mu\|_{\overline{\mathscr{C}}} := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \iint_{\widehat{B}} E(d\mu)\,d\mu \lesssim 1.$$

Proof. For any ball $B \subset \mathbb{R}^n$, we can estimate by Fubini's theorem that

$$\begin{aligned}
&\iint_{\widehat{B}} \Big[\frac{1}{|B(y,t)|}\int_{B(y,t)} \overline{\mathscr{A}}(d\mu)^{-1}(x)\,dx\Big]\,d\mu(y,t)\\
&\quad\simeq \iint_{\widehat{B}} \Big[\int_{B(y,t)} \overline{\mathscr{A}}(d\mu)^{-1}(x)\,dx\Big]\frac{d\mu(y,t)}{t^n}\\
&\quad= \int_{\mathbb{R}^n} \overline{\mathscr{A}}(d\mu)^{-1}(x)\Big[\iint_{\widehat{B}\cap\Gamma(x)} \frac{d\mu(y,t)}{t^n}\Big]\,dx\\
&\quad\le \int_B \overline{\mathscr{A}}(d\mu)^{-1}(x)\overline{\mathscr{A}}(d\mu)(x)\,dx = |B|.
\end{aligned}$$

Taking a supremum over balls $B \subset \mathbb{R}^n$ then proves the Carleson estimate. □

Remark 4.7.2. Denote by $\overline{\mathscr{V}}$ the class of bounded (signed and complex) Borel measures on $\overline{\mathbb{R}^{1+n}_+}$. Note that the above lemma also implies the factorization

$$\overline{\mathscr{V}} \to (T^1_\infty \cap C_{n.t.}) \cdot \overline{\mathscr{C}},$$

while the multiplication side $\overline{\mathscr{V}} \leftarrow (T^1_\infty \cap C_{n.t.}) \cdot \overline{\mathscr{C}}$ is just the Carleson's inequality (see [Ste93, p. 63] for example). Indeed, for $d\mu$ bounded on $\overline{\mathbb{R}^{1+n}_+}$,

$$|d\mu| = E(|d\mu|)^{-1} \cdot E(|d\mu|)|d\mu|$$

is then the desire factorization. First, using the lemma above, we have

$$\|E(|d\mu|)|d\mu|\|_{\overline{\mathscr{C}}} \lesssim 1.$$

And by (4.7.5), we see for any $(y,t) \in \mathbb{R}^{1+n}_+$ that

$$E(|d\mu|)^{-1}(y,t) \le \left(\frac{1}{|B(y,t)|}\int_{B(y,t)} \overline{\mathscr{A}}(|d\mu|)^{p_0}(x)dx\right)^{1/p_0}, 0<p_0<1.$$

Then for any $x \in \mathbb{R}^n$, we have

$$\mathscr{N}(E(|d\mu|)^{-1})(x) \le \mathbf{M}(\overline{\mathscr{A}}(|d\mu|)^{p_0})^{1/p_0}(x),$$

and by Lebesgue's theorem $E(|d\mu|)^{-1} \in C_{n.t.}$. By maximal theorem, we also have $E(|d\mu|)^{-1} \in T^1_\infty$, with the factorization estimate

$$\|E(|d\mu|)^{-1}\|_{T^1_\infty} \lesssim \|\overline{\mathscr{A}}(|d\mu|)\|_{L^1} \simeq |\mu|\big(\overline{\mathbb{R}^{1+n}_+}\big).$$

Remark 4.7.3. In F_1), the case $r_0=\infty$ is trivial. Suppose $0<r_0<\infty$ and $\mathscr{W}_{r_0}(u) \in C_{n.t.}$. As the constructed $v=\mathscr{W}^*_{r_0}(u)$ is continuous and satisfies $\mathscr{W}^*_\infty(v) \lesssim \mathscr{W}_{r_0}(u)$, we have $\mathscr{W}^*_\infty(v) \in C_{n.t.}$ after using the fact (4.7.1)

$$\lim_{\Gamma(x)\ni(y,t)\to x} W^*(y,t) = \lim_{\Gamma(x)\ni(y,t)\to x} W(y,t) = x, \forall\, x \in \mathbb{R}^n,$$

and the dominated convergence theorem.

In F_2), if $0<\max(p_0,q_0)<\infty$, we can also verify that $\mathscr{W}_\infty(v)$ is continuous in $\mathbb{R}^{1+n}_+$ and has the property of non-tangential convergence. In fact,

$$v^{\tilde{p}}(y,t) = |B(y,t)|^{-1}\int_{B(y,t)} \tilde{u}^{\tilde{p}}(x)dx, \forall\,(y,t) \in \mathbb{R}^{1+n}_+,$$

where $\tilde{u} \in L^{p_0}$ and $p_0 > \tilde{p}$. Then $v \in C_{n.t.}$ follows from Lebesgue's theorem. As

$$v(y,\alpha_* t) \lesssim \inf_{(z,s)\in W(y,t)} v(z,s) \le \sup_{(z,s)\in W(y,t)} v(z,s) \lesssim v(y,\alpha^* t)$$

hold true for any $(y,t) \in \mathbb{R}^{1+n}_+$, we then have

$$\mathscr{W}_\infty(v) = \sup_{(z,s)\in W(y,t)} v(z,s) \in C_{n.t.},$$

which is an easy consequence of the dominated convergence theorem. In all, the constructed factorization v is in $(T^{p_0,\infty}_\infty \cap C_{n.t.}) = (T^{p_0}_\infty \cap C_{n.t.})$.

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5

Weighted conical maximal regularity estimates for perturbed first order Dirac operators and application to Cauchy non-integral formulas

Abstract

This chapter can be roughly considered as a harmonic-analysis-side continuation of the recent memoir [AS14a] — P. Auscher and S. Stahlhut, *A priori estimates for boundary value elliptic problems via first order systems*, March 2014 — in the case of t-**dependent** coefficients for the elliptic systems. Its main purpose is to extrapolate in a conical way, namely, to tent spaces, the **a priori** weighted maximal regularity estimates obtained by P. Auscher and A. Axelsson in [AA11b]. The non-tent-space a priori estimates, together with the PDE-side extrapolation issues of [AA11b], namely, the extrapolation results on the representation of weak solutions and the solvability of boundary value elliptic problems will be addressed elsewhere.

For example, we prove in this chapter that the maximal regularity operator associated to the perturbed first order Dirac operator B_0D, defined formally by

$$F \mapsto \int_0^t B_0 D e^{-(t-s)|B_0D|}\chi^+(B_0D)F_s\,ds - \int_t^\infty B_0 D e^{-(s-t)|B_0D|}\chi^-(B_0D)F_s\,ds,$$

is bounded on a scale of weighted elliptic tent spaces $T^p_\beta\left(\mathbb{R}^{1+n}_+;\mathbb{C}^{(1+n)m}\right)$ for

$$p_- < p < p_+ \quad \text{and} \quad \beta \in (-1,1),$$

and modulo certain tent space boundedness of $\chi^\pm(DB_0)$, for

$$\max\left\{\frac{np_-}{n+\frac{1-\beta}{2}p_-},1\right\} < p < p_+ \quad \text{and} \quad \beta\in(-1,1),$$

with m the number of equations, (p_-,p_+) being the interval of $p\in(1,\infty)$ for which B_0D admits a bounded holomorphic functional calculus in $L^p\left(\mathbb{R}^n;\mathbb{C}^{(1+n)m}\right)$, $\chi^\pm(B_0D)$ being the spectral projections of the bisectorial operator B_0D, and $|B_0D| = \operatorname{sgn}(B_0D)B_0D = B_0D\left(\chi^+(B_0D)-\chi^-(B_0D)\right)$ being the generator of the $L^2\left(\mathbb{R}^n;\mathbb{C}^{(1+n)m}\right)$ semigroup $\left\{e^{-t|B_0D|}\right\}_{t>0}$. In the endpoint case $\beta=-1$, we show how to apply the Carleson duality results from [HR13] and the Whitney tent space framework of Chapter 4 to obtain similar maximal regularity estimates.

The twisted spectral projection $\operatorname{sgn}(B_0D)$, which is closely related to the Kato square root problem solved by P. Auscher et al., causes the main difficulty in analyzing the above maximal regularity operator. At the very technical side, our arguments reveal that L^2 theory of the Dirac operators is sufficient in estimating the singular part pertaining to the maximal regularity operator. The restrictions on the extrapolation interval of p come from the regular part. More precisely, for the singular part we make use of the tent space extrapolation (L^2-theory) of P. Auscher et al. in [AMP12], and for the regular part, we make use of the tent space extrapolation (L^p-theory) of P. Auscher et al. in [AKMP12] and further develop the extrapolation techniques designed in Chapter 3.

These a priori weighted conical maximal regularity results for first order systems have impacts on certain Cauchy non-integral formulas which allow us to construct some weak solutions to t-dependent elliptic systems. This way we extend the extrapolation picture of the 2012 El Escorial survey [Ros14] of A. Rosén.

Contents

5.1 Introduction

Some preliminaries are recalled in order. These materials are needed to give a rigorous account of the maximal regularity operators and their mapping properties.

a) We first define a scale of weighted elliptic tent spaces.

Let $\beta \in \mathbb{R}$, the real numbers, and let $m, n \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}$, the integers not less than 1. Set $N = (1+n)m$. Let $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}^{1+n}_+ = \mathbb{R}_+ \times \mathbb{R}^n$. Denote by (t,y), with $t>0$ and $y \in \mathbb{R}^n$, the points in $\mathbb{R}^{1+n}_+$, and by $B(y,t)$ the balls in $\mathbb{R}^n$. Let $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ be the set of locally square integrable complex $\mathbb{C}^N = \mathbb{C}^{(1+n)m}$ matrix valued functions in $\mathbb{R}^{1+n}_+$. For $0<p<\infty$, let $L^p = L^p\left(\mathbb{R}^n;\mathbb{C}^N\right)$ be the p-integrable Lebesgue space in $\mathbb{R}^n$, with the quasi-norm simply denoted by $\|\cdot\|_p$.

For $0<p<\infty$, define T^p_β as the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ functions such that

$$\|F\|_{T^p_\beta} := \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{1+n}_+}\frac{\mathbf{1}_{B(x,t)}(y)}{t^n}|F(t,y)|^2 t^\beta\, dt\, dy\right)^{p/2} dx\right)^{1/p} < \infty.$$

Let $\alpha \ge 0$. Define $T^{\infty,\alpha}_\beta$ as the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ functions such that

$$\|F\|_{T^{\infty,\alpha}_\beta} := \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(r^{-(n+2\alpha)}\iint_{(0,r)\times B(x,r)}|F(t,y)|^2 t^\beta\, dt\, dy\right)^{1/2} < \infty.$$

By Fubini's theorem,

$$T^2_\beta \simeq L^2\left(\mathbb{R}_+, t^\beta dt; L^2\left(\mathbb{R}^n;\mathbb{C}^N\right)\right)$$

for whatever β. In these tent space notations we omit β if $\beta=-1$, and omit α if $\alpha=0$. Note that T^p, $0<p\le\infty$, is the scale of classical tent spaces introduced in [CMS85].

One has the dual space identification $\left(T^q_\beta\right)' = T^{q'}_{-\beta}$ for $q \in (1,\infty)$, and $\left(T^q_\beta\right)' = T^{\infty, n\left(\frac{1}{q}-1\right)}_{-\beta}$ for $q \in (0,1]$, with the duality pairing given by

$$\langle F, G\rangle := \iint_{\mathbb{R}^{1+n}_+} F(t,y)\overline{G(t,y)}\, dt\, dy. \tag{5.1.1}$$

Note that this pairing differs with the one used in [CMS85].

Consider the modified non-tangential maximal functional: for $x \in \mathbb{R}^n$

$$\widetilde{N}_*(F)(x) := \sup_{t>0}\left(\frac{1}{t^{n+1}}\iint_{W(t,x)} |F(s,y)|^2 ds dy\right)^{1/2},$$

where $W(t,x) = (t/2, 2t) \times B(x,t)$ is called a Whitney box. Consider also the related Carleson type functional : for $x \in \mathbb{R}^n$

$$\widetilde{C}(E)(x) := \sup_{\substack{(t,y)\in\mathbb{R}^{1+n}_+ \\ B(y,t)\ni x}}\left(\frac{1}{t^n}\iint_{(0,t)\times B(y,t)}\left(\operatorname*{ess\,sup}_{W(s,z)}|E|\right)^2 \frac{dsdz}{s}\right)^{1/2}.$$

Let $\widetilde{T}^p$, $0 < p < \infty$, be the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ functions such that

$$\|F\|_{\widetilde{T}^p} := \left\|\widetilde{N}_*(F)\right\|_p < \infty.$$

Let $\widetilde{T}^\infty$ be the space of all $L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ functions such that

$$\|E\|_{\widetilde{T}^\infty} := \|\widetilde{C}(E)\|_\infty < \infty.$$

The functionals $\widetilde{N}_*$ and $\widetilde{C}$ are respectively introduced in [KP93] and [Dah86].

The above Whitney type tent spaces provide a framework in which we will work. For further information about this framework, see Chapter 4.

b) Next we review the functional calculus objects of perturbed Dirac operators.

Recall the first order Dirac operator

$$D := \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix} \text{ and } B_0 \in L^\infty\left(\mathbb{R}^n;\mathscr{L}\left(\mathbb{C}^N\right)\right),$$

namely, B_0 is a bounded complex matrix valued function in $\mathbb{R}^n$. Let

$$\mathscr{H} := \mathbf{N}_2(\operatorname{curl}_x) = \left\{f \in L^2\left(\mathbb{R}^n;\mathbb{C}^N\right) : \operatorname{curl}_x f = 0\right\} = \overline{\mathbf{R}_2(D)},$$

where $\overline{\mathbf{R}_2(D)}$ is the closure of the range of D in L^2, with the closure in the L^2 topology. Assume B_0 is also strictly accretive on $\overline{\mathbf{R}_2(D)}$, i.e., there exists some $\kappa > 0$ such that

$$\operatorname{Re}\langle f, B_0 f\rangle \gtrsim \kappa\|f\|_2, \quad \forall f \in \overline{\mathbf{R}_2(D)}, \tag{5.1.2}$$

where $\langle\cdot,\cdot\rangle$ is the L^2 sesquilinear inner product.

Define closed and open sectors and double sectors in the complex plane by

$$\overline{S_{\omega+}} := \{\lambda \in \mathbb{C} : |\arg\lambda| \le \omega\} \cup \{0\}, \qquad \overline{S_\omega} := \overline{S_{\omega+}} \cup \left(-\overline{S_{\omega+}}\right),$$

$$S_{\nu+} := \left\{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \nu\right\}, \qquad S_\nu := S_{\nu+} \cup (-S_{\nu+}),$$

and define the angle of accretivity of B_0 to be

$$\omega := \sup_{f \neq 0, f \in \overline{\mathbf{R}_2(D)}} \left|\arg\left(\langle f, B_0 f\rangle\right)\right|.$$

Note that by the accretivity assumption (5.1.2), $\omega < \pi/2$.

Throughout this chapter, let T be one of the closed operators $\{DB_0, B_0D\}$. It was proved in [AAM10a] that the operator T has a bounded holomorphic functional calculus in L^2. More precisely, for each bounded holomorphic function b on a double sector S_ν, $\omega < \nu < \pi/2$, there is a uniquely defined operator $b(T)$ which is bounded in L^2, and these uniquely defined operators enjoy the functional calculus bound

$$\|b(T)\|_{L^2 \to L^2} \lesssim \|b\|_{L^\infty(S_\nu)}.$$

Here b can be the characteristic functions χ^+ and χ^- for the right and left open half planes. For background materials on holomorphic functional calculi, see [ADM96].

c) Finally we define the maximal regularity operators in question.

Let $|T| = T\left(\chi^+(T) - \chi^-(T)\right)$. Consider the maximal regularity operator

$$\mathbf{M}_T(\mathbf{E}F)_t = \int_0^t Te^{-(t-s)|T|}\chi^+(T)(\mathbf{E}F)_s ds - \int_t^\infty Te^{-(s-t)|T|}\chi^-(T)(\mathbf{E}F)_s ds,$$

defined for $\mathbf{E}F \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$, where $L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ is the class of $L^2\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ functions having compact support in $\mathbb{R}^{1+n}_+$, and it is dense in T^p for $0 < p < \infty$ (see [CMS85]). Here $\mathbf{E} \in L^\infty\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$ is a pointwise multiplication on $F \in L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. A rigorous formulation of $\mathbf{M}_T$ needs the holomorphic operational calculus in [AFM98, AA11b]. This functional/operational calculus information involved in defining $\mathbf{M}_T$ also explains the difference with the maximal regularity operators considered in Part II.

Consider a variant of $\mathbf{M}_T$, defined by

$$\mathbf{S}_{DB_0}(\mathbf{E}F)_t = \int_0^t e^{-(t-s)|DB_0|}\chi^+(DB_0)D(\mathbf{E}F)_s ds - \int_t^\infty e^{-(s-t)|DB_0|}\chi^-(DB_0)D(\mathbf{E}F)_s ds,$$

on $\mathbf{E}F \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. The reason for studying $\mathbf{S}_{DB_0}$ is given in Section 5.7. Using the L^2 functional calculus relation between DB_0 and B_0D,

$$b(DB_0)D = Db(B_0D),$$

we can also rewrite

$$\begin{aligned}\mathbf{S}_{DB_0}(\mathbf{E}F)_t &= \int_0^t De^{-(t-s)|B_0D|}\chi^+(B_0D)(\mathbf{E}F)_s ds \\ &\qquad - \int_t^\infty De^{-(s-t)|B_0D|}\chi^-(B_0D)(\mathbf{E}F)_s ds \\ &=: \mathbf{S}^+_{DB_0}(\mathbf{E}F)_t - \mathbf{S}^-_{DB_0}(\mathbf{E}F)_t,\end{aligned} \tag{5.1.3}$$

when $\mathbf{E}F \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$.

Now we can describe the motivation and the main result of this chapter.

Let $|\beta| \le 1$. It is proved in [AA11b, Ros14] that $\mathbf{S}_{DB_0}$ (or $\mathbf{M}_{DB_0}$ and $\mathbf{M}_{B_0D}$) extends to a bounded operator on T^2_β for $|\beta| < 1$ and $\mathbf{E} \in L^\infty$, and the same statement holds for $\beta = 1$ with $\mathbf{E} \in L^\infty$ replaced by $\mathbf{E} \in \widetilde{T}^\infty$, and for $\beta = -1$ with $\mathbf{E} \in L^\infty$ replaced by $\mathbf{E} \in \widetilde{T}^\infty$ and with T^2 replaced by $\widetilde{T}^2$. The aim of this chapter is to extrapolate to tent spaces these weighted maximal regularity estimates, and use the recently developed Carleson duality results in [HR13] and Chapter 4 to deal with endpoint weights $\beta = \pm 1$.

Recall from [AS14b] that there is an interval (p_-, p_+) of $p \in (1,\infty)$ for which DB_0 and B_0D admit a bounded holomorphic functional calculus in $L^p\left(\mathbb{R}^n;\mathbb{C}^N\right)$. Let $\widetilde{p}_- = (p_+)'$ and $\widetilde{p}_+ = (p_-)'$, where as usual, for $q \in (1,\infty)$, q' is given by $1/q' + 1/q = 1$. The interval $(\widetilde{p}_-, \widetilde{p}_+)$ is the corresponding interval of $p \in (1,\infty)$ for which DB_0^* and B_0^*D admit a bounded holomorphic functional calculus in $L^p\left(\mathbb{R}^n;\mathbb{C}^N\right)$.

Our main result reads as follows.

Theorem 5.1.1. *We have the following tent space maximal regularity results.*

1) Intermediate-weight maximal regularity estimates. Assume $\mathbf{E} \in L^\infty$.

(i) For $\beta \in (-1,1)$ *and* $p_- < p < p_+$ *we have*

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{T^p_\beta} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_\beta}. \tag{5.1.4}$$

(ii) For $\beta \in (-1,1)$ *and* $\widetilde{p}_- < \widetilde{p} < \widetilde{p}_+$ *we have*

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'} \lesssim \|\mathbf{E}\|_\infty \|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}. \tag{5.1.5}$$

2) Endpoint-weight maximal regularity estimates. Assume $\mathbf{E} \in \widetilde{T}^\infty$.

(i) For $p_- < p < p_+$ we have

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{\widetilde{T}^p} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p}. \tag{5.1.6}$$

(ii) For $\widetilde{p}_- < \widetilde{p} < \widetilde{p}_+$ we have

$$\left\|\mathbf{S}_{DB_0}(\mathbf{E}F)\right\|_{(T^{\widetilde{p}})'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{(T^{\widetilde{p}})'}. \tag{5.1.7}$$

In the above theorem, the tent space duality is with respect to the pairing (5.1.1). Also 2)-(i) corresponds to 1)-(i) for $\beta = -1$ and 2)-(ii) corresponds to 1)-(ii) for $\beta = 1$.

We end this introduction with several comments.

C1) Note that we have the formal operational relations:

$$\mathbf{S}_{DB_0} = \mathbf{M}_{DB_0}B_0^{-1} \text{ and } \mathbf{S}_{DB_0} = B_0^{-1}\mathbf{M}_{B_0D},$$

if we assume $B_0^{-1} \in L^\infty$. For this reason, we also view $\mathbf{S}_{DB_0}$ as a maximal regularity operator. In application to boundary value elliptic problems (see Section 5.7 below), the assumption $B_0^{-1} \in L^\infty$ is always valid for second order equations, but not for systems.

C2) We point out that the claim in part 1)-(ii) of Theorem 5.1.1 follow (at least formally), as we will see, by duality arguments from the corresponding claim in part 1)-(i). In the sequel we mean, if not specified, the proof of Theorem 5.1.1 by the arguments leading directly to claims in part 1)-(i) and part 2).

C3) The proof of Theorem 5.1.1 uses a more complicated version of the extrapolation theory presented in Chapter 3. This is also the most subtle part of our arguments.

C4) We also point out in part 1) of Theorem 5.1.1 for $p = 2$ (or $\widetilde{p} = 2$) the assumption $B_0^{-1} \in L^\infty$ is used in the survey article of A. Rosén [Ros14]. We remove this assumption by using the duality in the trace spaces in a different way.

C5) In Section 5.6 we give certain conditional extensions of Theorem 5.1.1 outside the functional calculus intervals.

5.2 Complements on functional calculus

Recall the first order Dirac operator $D = \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}$ and $B_0 \in L^\infty\left(\mathbb{R}^n; \mathscr{L}\left(\mathbb{C}^N\right)\right)$. In this section we complement several aspects on the functional calculus of DB_0 and B_0D, which is given in [AS14b, AS14a] (see also [Aji07, HMP11, HM10]), and the related $L^q - L^r$ theory (boundedness and off-diagonal decay) which is given in [Sta14].

It is shown in [AS14b] that there exists an open interval (p_-, p_+) in $(1, \infty)$ for which the bounded holomorphic functional calculus of B_0D holds in L^p, and for those p, the operator B_0D in L^p has natural domain

$$\mathbf{D}_p(B_0D) = \left\{u \in L^p : Du \in L^p \text{ and } B_0Du \in L^p\right\}.$$

The interval for DB_0 is the same, with domain

$$\mathbf{D}_p(DB_0) = \{f \in L^p : B_0 f \in L^p \text{ and } DB_0 f \in L^p\}.$$

Let $\overline{\mathbf{R}_p(T)}$ be the closure of the range of T in L^p, $p_- < p < p_+$, with the closure in the L^p topology, and let $\mathbf{N}_p(T)$ be the kernel or null space of T in L^p.

For $B_0 = I$, the interval (p_-, p_+) for $D = DB_0 = B_0 D$ is $(1, \infty)$. Consider the orthogonal projection $\mathbb{P}$ from L^2 onto $\overline{\mathbf{R}_2(D)}$ along $\mathbf{N}_2(D)$, and the projection $\mathbb{P}_{B_0 D}$ from L^2 onto $\overline{\mathbf{R}_2(B_0 D)}$ along $\mathbf{N}_2(B_0 D)$. Note that $\mathbf{N}_2(B_0 D) = \mathbf{N}_2(D)$ by the accretivity of B_0. Hence

$$\langle \mathbb{P}_{B_0 D} u, v \rangle = \langle u, v \rangle, \quad \forall\, u \in L^2, \quad \forall\, v \in \overline{\mathbf{R}_2(D)}. \tag{5.2.1}$$

Lemma 5.2.1. *For $p \in (p_-, p_+)$, $B_0|_{\overline{\mathbf{R}_p(D)}} : \overline{\mathbf{R}_p(D)} \to \overline{\mathbf{R}_p(B_0 D)}$ is an isomorphism.*

For $p \in (p_-, p_+)$, $\mathbb{P}$ extends to an isomorphism between $\overline{\mathbf{R}_p(B_0 D)}$ and $\overline{\mathbf{R}_p(D)}$, with its inverse given by $\mathbb{P}_{B_0 D} : \overline{\mathbf{R}_p(D)} \to \overline{\mathbf{R}_p(B_0 D)}$.

These results are proved in [AS14a, Lemma 3.4, Proposition 3.8].

Remark 5.2.2. If B_0 is invertible in L^∞, then the multiplication by B_0 is invertible in $\mathscr{L}(L^p)$ for all $1 < p < \infty$, and the inverse of $B_0|_{\overline{\mathbf{R}_p(D)}} : \overline{\mathbf{R}_p(D)} \to \overline{\mathbf{R}_p(B_0 D)}$ is given by $B_0^{-1}|_{\overline{\mathbf{R}_p(B_0 D)}} : \overline{\mathbf{R}_p(B_0 D)} \to \overline{\mathbf{R}_p(D)}$.

5.2.1 R-boundedness

Let X be a Banach space. A set $\tau \subset \mathscr{L}(X)$ is called R-bounded in X, if there is a constant C such that for all $T_1, \cdots, T_n \in \tau$, $x_1, \cdots, x_n \in X$ and $n \in \mathbb{N}^*$

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j(x_j) \right\|_X^2 du \le C \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|_X^2 du, \tag{5.2.2}$$

where r_j is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$, e. g. the Rademacher functions $r_j(t) = \operatorname{sign}\left(\sin\left(2^j \pi t\right)\right)$.

By definition, a single operator which is bounded in a Banach space X is always R-bounded in X. When X is an $L^p(\mathbb{R}^n)$ subspace, $1 < p < \infty$, the R-boundedness of $\{T(t)\}_{t>0} \subset \mathscr{L}(X)$ is equivalent to

$$\left\| \left(\int_0^\infty |(T(t)F_t)(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_X \lesssim \left\| \left(\int_0^\infty |F_t(\cdot)|^2 \frac{dt}{t} \right)^{1/2} \right\|_X \tag{5.2.3}$$

for all F such that the right hand side is finite. Let $H = L^2\left(\mathbb{R}_+, t^{-1} dt\right)$. Hence, when X is an $L^p(\mathbb{R}^n)$ subspace, for F such that the right hand side of (5.2.3) is finite, we write $F \in X(\mathbb{R}^n; H)$. Note that in (5.2.3) one can replace H by $L^2\left(\mathbb{R}_+, t^\beta dt\right)$ for $\beta \in \mathbb{R}$.

We summarize the main vector-valued boundedness results that we will use later. They are direct consequences of the bounded holomorphic functional calculus.

Lemma 5.2.3. *For $p \in (p_-, p_+)$, $\chi^\pm(T)$ are $\overline{\mathbf{R}_p(T)}(\mathbb{R}^n; H)$-bounded, $\{e^{-t|T|}\}_{t>0}$ is R-bounded in $\overline{\mathbf{R}_p(T)}$, and $\mathbb{P}_{B_0 D}$ maps $L^p(\mathbb{R}^n; H)$ bounded into $\overline{\mathbf{R}_p(B_0 D)}(\mathbb{R}^n; H)$.*

We point out that in [HM10, AS14b] the R-boundedness of the resolvents of T is used to characterize its bounded holomorphic functional calculus in L^p, $1 < p < \infty$. The R-boundedness of the semigroup $\{e^{-t|T|}\}_{t>0}$ follows from the one for the resolvents through vector-valued Laplace transform (see [Wei01, Theorems 2.10 and 4.2]).

5.2.2 Local coercive inequalities

We first state the following local coercivity inequalities on balls.

Lemma 5.2.4. *Let $p_- < q < p_+$. For any $u \in L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^N)$ with $Du \in L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^N)$, any ball $B(x,r)$ in $\mathbb{R}^n$ and $c > 1$, we have*

$$\int_{B(x,r)} |Du|^q \lesssim \int_{B(x,cr)} |B_0 Du|^q + r^{-q} \int_{B(x,cr)} |u|^q, \tag{5.2.4}$$

with the implicit constants depending only on B_0, n, N and c.

The proof is entirely similar to the one for [AS14a, Lemma 5.14], which corresponds to the case $q = 2$ here. We include this argument for completeness.

Proof. Let χ be a scalar-valued cut-off function with $\chi = 1$ on $B(x,r)$, supported in $B(x,cr)$ and with $\|\nabla\chi\|_\infty \lesssim r^{-1}$. As $\chi u \in \mathbf{D}_q(D)$ and using that the commutator between χ and D is the pointwise multiplication by a matrix with bound controlled by $|\nabla\chi|$,

$$\int_{B(x,r)} |Du|^q \le \int_{\mathbb{R}^n} |\chi Du|^q \lesssim \int_{\mathbb{R}^n} |D(\chi u)|^q + \int_{B(x,cr)} |\nabla\chi|^q |u|^q.$$

By Lemma 5.2.1, as $p_- < q < p_+$ we have

$$\int_{\mathbb{R}^n} |D(\chi u)|^q \lesssim \int_{\mathbb{R}^n} |B_0 D(\chi u)|^q.$$

Then, we use again the commutation between χ and D together with $\|B_0\|_\infty$.

This proves the estimate (5.2.4). □

These inequalities shall motivate part of the results in next subsection.

5.2.3 Off-diagonal decay

Let $\langle x\rangle = 1 + |x|$. We recall the following measure of decay on operator families.

Definition 5.2.5. Let $1 \le q \le r \le \infty$.

1). A family of L^2 bounded operators $\{T(t)\}_{t>0}$ is said to satisfy the $L^q - L^r$ off-diagonal decay, with decay order $K > 0$, if there holds

$$\|\mathbf{1}_E T(t) \mathbf{1}_F u\|_{L^r} \lesssim t^{\frac{n}{r}-\frac{n}{q}} \langle \operatorname{dist}(E,F)/t\rangle^{-K} \|u\|_{L^q}$$

for all closed sets $E, F \subset \mathbb{R}^n$, all $t > 0$ and all $u \in L^q \cap L^2$ with support in F.

2). A family of L^2 bounded operators $\{T(t)\}_{t>0}$ is said to be $L^q - L^r$ bounded if it satisfies the $L^q - L^r$ off-diagonal decay in the case $E = F = \mathbb{R}^n$.

Using ideas from Lemma 5.2.4 we show

Lemma 5.2.6. *Let $q \in (p_-, p_+)$ and $M \in \mathbb{N}^*$. For every $K > 0$ there exists $C_K > 0$ such that*

$$\left\|\mathbf{1}_E tD(I + itB_0D)^{-M} \mathbf{1}_F u\right\|_{L^q(E)} \le C_K \langle \operatorname{dist}(E,F)/t\rangle^{-K} \|u\|_{L^q(F)} \tag{5.2.5}$$

for all closed sets $E, F \subset \mathbb{R}^n$, all $t \neq 0$ and all $u \in L^q \cap L^2$ with support in F.

Proof. As observed in [HM10, Lemma 2.4], for $q \in (p_-, p_+)$ the $L^q - L^q$ off-diagonal decay for $(I + itB_0D)^{-1}$ is a consequence of its L^q boundedness. Also the off-diagonal decay (with any decay order) composes. We now prove the lemma for $M = 1$.

Let $R_t^{B_0} = (I + itB_0D)^{-1}$. Let $d = \operatorname{dist}(E,F)$, where E and F are two closed sets in $\mathbb{R}^n$ as given. The uniform L^q boundedness for $tDR_t^{B_0}$ can be easily verified by Lemma 5.2.1 and functional calculus of B_0D, thus it reduces to the case $|t| \le \alpha d$ for some small constant $\alpha > 0$ to be chosen.

Assume $u \in L^q$ with $\operatorname{supp} u \subset F$. Set $v = R_t^{B_0} u = \mathbf{D}_q(D) \cap L^q$. First we construct

$$\widetilde{E} := \left\{x \in \mathbb{R}^n \mid \operatorname{dist}(x,E) < \operatorname{dist}(x,F)/2\right\}.$$

Let χ be a scalar-valued cut-off function with $\chi = 1$ on E, supported in $\widetilde{E}$ and with $\|\nabla\chi\|_\infty \lesssim d^{-1}$. As $\chi v \in \mathbf{D}_q(D)$ and using that the commutator between χ and D is the pointwise multiplication by a matrix with bound controlled by $|\nabla\chi|$,

$$\int_E |Dv|^q \le \int_{\mathbb{R}^n} |\chi Dv|^q \lesssim \int_{\mathbb{R}^n} |D(\chi v)|^q + \int_{\widetilde{E}} |\nabla\chi|^q |v|^q.$$

By Lemma 5.2.1, as $q \in (p_-, p_+)$ we have

$$\int_{\mathbb{R}^n} |D(\chi v)|^q \lesssim \int_{\mathbb{R}^n} |B_0 D(\chi v)|^q.$$

Now, we use again the commutation between χ and D together with $\|B_0\|_\infty$. This shows

$$\left\|tDR_t^{B_0} u\right\|_{L^q(E)} \lesssim \left\|tB_0DR_t^{B_0} u\right\|_{L^q(\widetilde{E})} + \frac{t}{d}\left\|R_t^{B_0} u\right\|_{L^q(\widetilde{E})}.$$

Observe that $tDR_t^{B_0} u = -i\left(u - R_t^{B_0} u\right)$. The lemma follows from the off-diagonal estimates for $R_t^{B_0}$ proved in [AAM10a, Proposition 5.1] for $q = 2$ and observed in [HM10, Lemma 2.4] for $q \in (p_-, p_+)$. □

The property in the following lemma is often called hyperboundedness.

Lemma 5.2.7. *Let $p_- < q \leq r < p_+$ and $\alpha, M \in \mathbb{N}^*$ with $M - \alpha > n\left(\frac{1}{q} - \frac{1}{r}\right)$. Then*

$$\left\|(tT)^\alpha (I + itT)^{-M} u\right\|_r \lesssim t^{\frac{n}{r} - \frac{n}{q}} \|u\|_q \tag{5.2.6}$$

for all $u \in L^q \cap L^r$ (by density for all $u \in L^q$).

Proof. We follow Claim 3.5 of [Sta14].

Suppose $r \in (p_-, p_+)$ and $q \in [r_*, r] \cap (p_-, p_+)$. Here $r_* = \frac{nr}{n+r}$.

We first consider estimates for $T_{B_0 D} = (B_0 D)^\alpha (I + iB_0 D)^{-M}$. We have

$$T_{B_0 D} : \overline{\mathbf{R}_q(B_0 D)} \to \overline{\mathbf{R}_q(B_0 D)}$$

and, since $\alpha + 1 \leq M$,

$$T_{B_0 D} : \overline{\mathbf{R}_q(B_0 D)} \to \mathbf{D}_q(B_0 D).$$

As $\mathbf{D}_q(B_0 D) = \mathbf{D}_q(D)$ by Remark 2.8 of [Sta14], we deduce

$$\mathbb{P}_{\overline{\mathbf{R}_q(D)}} T_{B_0 D} : \overline{\mathbf{R}_q(B_0 D)} \to \overline{\mathbf{R}_q(D)} \cap \mathbf{D}_q(D) \subset W^{1,q}.$$

Thus, by Sobolev embedding theorem, we obtain for all $u \in \overline{\mathbf{R}_q(B_0 D)}$ that

$$\left\|\mathbb{P}_{\overline{\mathbf{R}_q(D)}} T_{B_0 D} u\right\|_{L^r} \lesssim \|u\|_{L^q}.$$

If u is also in $\overline{\mathbf{R}_r(B_0 D)}$ then $T_{B_0 D} u \in \overline{\mathbf{R}_r(B_0 D)}$. By Remark 2.8 of [Sta14] for D we have $\mathbb{P}_{\overline{\mathbf{R}_q(D)}} = \mathbb{P}_{\overline{\mathbf{R}_r(D)}}$ on $L^q \cap L^r$, hence on $\overline{\mathbf{R}_q(D)} \cap \overline{\mathbf{R}_r(D)}$. From this we deduce

$$\mathbb{P}_{\overline{\mathbf{R}_q(D)}} T_{B_0 D} u = \mathbb{P}_{\overline{\mathbf{R}_r(D)}} T_{B_0 D} u \in \overline{\mathbf{R}_q(D)} \cap \overline{\mathbf{R}_r(D)}$$

for all $u \in \overline{\mathbf{R}_q(B_0 D)} \cap \overline{\mathbf{R}_r(B_0 D)}$. Since $\mathbb{P}_{\overline{\mathbf{R}_r(D)}} : \overline{\mathbf{R}_r(B_0 D)} \to \overline{\mathbf{R}_r(D)}$ is an isomorphism by Lemma 5.2.1, we get

$$\begin{aligned} \|T_{B_0 D} u\|_{L^r} &\lesssim \left\|\mathbb{P}_{\overline{\mathbf{R}_r(D)}} T_{B_0 D} u\right\|_{L^r} \\ &= \left\|\mathbb{P}_{\overline{\mathbf{R}_q(D)}} T_{B_0 D} u\right\|_{L^r} \lesssim \|u\|_{L^q} \end{aligned}$$

for all $u \in \overline{\mathbf{R}_q(B_0 D)} \cap \overline{\mathbf{R}_r(B_0 D)}$.

Since we know that $B_0 : \overline{\mathbf{R}_q(D)} \to \overline{\mathbf{R}_q(B_0 D)}$ and $B_0 : \overline{\mathbf{R}_r(D)} \to \overline{\mathbf{R}_r(B_0 D)}$ are isomorphisms, thus the similarity property in Remark 2.10 of [Sta14] yields

$$\|T_{DB_0} u\|_{L^r} \lesssim \|u\|_{L^q}, \quad T_{DB_0} = (DB_0)^\alpha (I + iDB_0)^{-M},$$

for all $u \in \overline{\mathbf{R}_q(D)} \cap \overline{\mathbf{R}_r(D)}$. Now, we use a rescaling argument of [Sta14] and note that for B_0^t defined by multiplication of $B_0^t(x) := B_0(tx)$ has the same properties as B_0 with uniform bounds in t. Let $u_t(x) := u(tx)$. Then we have as above

$$\left\|\left(DB_0^t\right)^\alpha \left(I + iDB_0^t\right)^{-M} u_t\right\|_{L^r} \lesssim \|u_t\|_{L^q}$$

and substitution $tx \mapsto x$ yields the estimate

$$\left\|(tDB_0)^{\alpha}\left(I+itDB_0\right)^{-M}u\right\|_{L^r} \lesssim t^{\frac{n}{r}-\frac{n}{q}}\|u\|_{L^q}.$$

for all $u \in \overline{\mathbf{R}_q(D)} \cap \overline{\mathbf{R}_r(D)}$. Using the decomposition from Lemma 2.13 of [Sta14]

$$\left[L^q \cap L^r\right] = \left[\overline{\mathbf{R}_q(D)} \cap \overline{\mathbf{R}_r(D)}\right] \oplus \left[\mathbf{N}_q(DB_0) \cap \mathbf{N}_r(DB_0)\right],$$

and the fact that $DB_0\mathbf{N}_q(DB_0) = DB_0\mathbf{N}_r(DB_0) = 0$ (since $\alpha \in \mathbb{N}^*$), we see (5.2.6) for DB_0 when $r \in \left(p_-, p_+\right)$ and $q \in [r_*, r] \cap \left(p_-, p_+\right)$.

The claim in the lemma for $T = DB_0$ and for general $p_- < q \le r < p_+$ follows by an iteration argument [1] in [Sta14, p. 9], which uses the critical condition $M-\alpha > n\left(\frac{1}{q}-\frac{1}{r}\right)$.

The lemma for $T = B_0D$ follows by similarity as in [Sta14]. We omit the details. □

Now we summarize in the following lemma the $L^q - L^r$ off-diagonal decay for the resolvent families of perturbed first order Dirac operators.

Lemma 5.2.8. *Let $p_- < q \le r < p_+$.*

1). Let $\alpha \in \mathbb{N}^*$, $M \in \mathbb{N}^*$ and $M-\alpha > n\left(\frac{1}{q}-\frac{1}{r}\right)$. Let $T(t) = (tT)^{\alpha}(I+itT)^{-M}$, $t > 0$. Then $\{T(t)\}_{t>0}$ has $L^q - L^r$ off-diagonal decay of arbitrary order, that is, for any $K > 0$, for all $t > 0$, closed sets $E, F \subset \mathbb{R}^n$ and $u \in L^q \cap L^2$ with support in F:

$$\|\mathbf{1}_E T(t)\mathbf{1}_F u\|_r \lesssim t^{\frac{n}{r}-\frac{n}{q}}\langle \operatorname{dist}(E,F)/t\rangle^{-K}\|u\|_q. \tag{5.2.7}$$

The implicit constants are independent of t, E, F and u.

2). The same results hold with $T(t)$ replaced by $tD(tB_0D)^{\alpha-1}(I+itB_0D)^{-M}$.

Of particular interest in part 2) of the above lemma is the case $\alpha = 1$ and $M = 2$.

Proof. The off-diagonal decay (5.2.7) is given as Example 3.7 in [Sta14]. Unlike his approach (using [Sta14, Claim 3.4]), we use Lemma 5.2.7 to have the $L^q - L^r$ boundedness of $\{T(t)\}_{t>0}$. As in [Sta14], the $L^q - L^r$ off-diagonal decay follows by interpolation of the $L^q - L^r$ boundedness with the $L^r - L^r$ off-diagonal decay.

For the claim on $sD(sB_0D)^{\alpha-1}(I+isB_0D)^{-M}$, with $\alpha, M \in \mathbb{N}^*$ and $M-\alpha > n\left(\frac{1}{q}-\frac{1}{r}\right)$, its $L^q - L^r$ boundedness follows from Lemma 5.2.1 and the $L^q - L^r$ boundedness of $(sB_0D)^{\alpha}(I+isB_0D)^{-M}$. We use Lemma 5.2.6 to have the $L^r - L^r$ off-diagonal decay for $sD(sB_0D)^{\alpha-1}(I+isB_0D)^{-M}$. We conclude the claim in 2) by interpolation. □

Recall that $H^{\infty}\left(S_{\mu}\right)$ is the space of bounded holomorphic functions in S_{μ}, with its norm simply denoted by $\|\cdot\|_{\infty}$. For $\sigma, \tau \ge 0$,

$$\Psi_{\sigma}^{\tau}\left(S_{\mu}\right) = \left\{\psi \in H^{\infty}\left(S_{\mu}\right) : \psi(z) = O\left(\inf\left\{|z|^{\sigma}, |z|^{-\tau}\right\}\right)\right\}.$$

That the above lemma is interesting lies in the following general claim on the $L^q - L^r$ off-diagonal decay for holomorphic functions of perturbed first order Dirac operators.

1. More precisely, this is the paragraph in [Sta14, p. 9] in deducing Claim 3.4 from Claim 3.6 there.

Lemma 5.2.9 ([Sta14])**.** *Let $p_- < q \le r < p_+$.*

Let $\psi \in \Psi_\sigma^\tau(S_\mu)$ with $\sigma > 0, \tau > \frac{n}{q} - \frac{n}{r}$ and $g \in H^\infty(S_\mu)$. Then for all $t > 0$, closed sets $E, F \subset \mathbb{R}^n$ and $u \in L^q \cap L^2$ with support in F:

$$\left\| \mathbf{1}_E g(T)\psi(tT)\mathbf{1}_F u \right\|_r \lesssim \|g\|_\infty t^{\frac{n}{r}-\frac{n}{q}} \langle \operatorname{dist}(E,F)/t\rangle^{-\sigma c} \|u\|_q. \tag{5.2.8}$$

Here, c is a positive number smaller than $1 - \left(\frac{1}{q} - \frac{1}{r}\right)\left(\frac{1}{p_-} - \frac{1}{p_+}\right)^{-1}$ and can be taken equal to 1 when $q = r$. The implicit constants are independent of t, E, F and u.

This off-diagonal decay is given in Proposition 3.3 in [Sta14], and the decay order can not be improved even for $B_0 = I$.

Lemma 5.2.10. *Let $p_- < q \le r < p_+$.*

For each integer $K > 0$ and each $\mu \in (\omega, \pi/2)$, there exists $\phi^\pm \in H^\infty(S_\mu)$ such that $\phi^\pm(sT)$ coincides with $e^{-s|T|}$ on $\chi^\pm(T)\overline{\mathbf{R}_2(T)}$, $\{\phi^\pm(sT)\}_{s>0}$ has $L^2 - L^2$ off-diagonal decay of order K, and $\left\{sT\phi^\pm(sT)\right\}_{s>0}$ has $L^q - L^r$ off-diagonal decay of order K.

Moreover, $\left\{sD\phi^\pm(sB_0D)\right\}_{s>0}$ has $L^2 - L^2$ and $L^q - L^r$ off-diagonal decay of order K.

Proof. We recall the construction. Let $M \in \mathbb{N}^*$ such that $M - 1 > n\left(\frac{1}{q} - \frac{1}{r}\right)$. For each $\ell \in \mathbb{N}^*$ one can find coefficients c_m with

$$\psi^+(z) := e^{-z} - \sum_{m=1}^{\ell} c_m (1 + imz)^{-M} = O\left(z^\ell\right)$$

near 0. For $z \in \mathbb{C}$, set

$$\phi^+(z) := \sum_{m=1}^{\ell} c_m (1 + imz)^{-M} + \psi^+(z)\chi^+(z).$$

One can easily see that $\phi^+ \in H^\infty(S_\mu)$. Picking (different) coefficients such that

$$\psi^-(z) = e^{z} - \sum_{m=1}^{\ell} \tilde{c}_m (1 + imz)^{-M} = O\left(z^\ell\right)$$

near 0, we get ϕ^-. Using these approximations, the off-diagonal decay in Lemma 5.2.10 is then a consequence of Lemma 5.2.8 and Lemma 5.2.9. □

5.3 Review of Hardy spaces

We recall some fundamental results for the Hardy space theory associated to perturbed first order Dirac operators established in [AS14a].

5.3.1 General theory

Recall that for $\sigma, \tau \geq 0$,

$$\Psi_\sigma^\tau\left(S_\mu\right) = \left\{\psi \in H^\infty\left(S_\mu\right) : \psi(z) = O\left(\inf\left(|z|^\sigma, |z|^{-\tau}\right)\right)\right\}.$$

So

$$\Psi\left(S_\mu\right) = \bigcup_{\sigma>0,\tau>0} \Psi_\sigma^\tau\left(S_\mu\right).$$

Let T be a bisectorial operator of type $\omega \in [0, \pi/2)$, namely, $\sigma(T) \subset \overline{S_\omega}$, and there are resolvent estimates

$$\left\|(\lambda I - T)^{-1}\right\| \lesssim 1/\mathrm{dist}(\lambda, S_\mu)$$

when $\lambda \notin S_\mu$, $\omega < \mu < \pi/2$. Let $\psi \in \Psi\left(S_\mu\right)$ with $\mu \in (\omega, \pi/2)$. Let $\mathbb{Q}_{\psi,T} = \psi(\cdot T)$, with $(\mathbb{Q}_{\psi,T} h)_t = \psi(tT)h$ for $h \in L^2$ or in the closure $\overline{\mathbf{R}_2(T)}$. Define the pre-Hardy and pre-Hölder spaces as

$$\mathbb{H}_{\psi,T}^p = \left\{h \in \overline{\mathbf{R}_2(T)} : \mathbb{Q}_{\psi,T} h \in T^p\right\}, \quad 0 < p < \infty, \tag{5.3.1}$$

and

$$\mathbb{L}_{\psi,T}^\alpha = \left\{h \in \overline{\mathbf{R}_2(T)} : \mathbb{Q}_{\psi,T} h \in T^{\infty,\alpha}\right\}, \quad \alpha \geq 0. \tag{5.3.2}$$

These spaces do not depend on $\psi \in \Psi_\sigma^\tau\left(S_\mu\right)$ provided σ and τ are large enough.

For $0 < \gamma$, let

$$\Psi^\gamma\left(S_\mu\right) = \bigcup_{\sigma>0,\tau>\gamma} \Psi_\sigma^\tau\left(S_\mu\right),$$

$$\Psi_\gamma\left(S_\mu\right) = \bigcup_{\sigma>\gamma,\tau>0} \Psi_\sigma^\tau\left(S_\mu\right).$$

Set $\gamma(p) = \left|\frac{n}{p} - \frac{n}{2}\right|$ for $0 < p \leq \infty$. If $p \leq 1$ and $\alpha = n\left(\frac{1}{p} - 1\right)$, then $\gamma(p) = \frac{n}{2} + \alpha$. Define $\mathbb{H}_T^p = \mathbb{H}_{\psi,T}^p$ if $\psi \in \Psi^{\gamma(p)}$ $(0 < p \leq 2)$, $\mathbb{H}_T^p = \mathbb{H}_{\psi,T}^p$ if $\psi \in \Psi_{\gamma(p)}$ $(2 \leq p < \infty)$, and $\mathbb{L}_T^\alpha = \mathbb{L}_{\psi,T}^\alpha$ if $\psi \in \Psi_{\gamma(p)}$ (with $\alpha = n\left(\frac{1}{p} - 1\right)$). It is given in [AS14a, Corollary 4.4] that this definition is stable with respect to ψ satisfying the corresponding decay conditions.

As usual, $H^p = L^p$ for $p > 1$.

Lemma 5.3.1. *We have the following results.*

Let $\frac{n}{n+1} < p < \infty$. Assume that $\mathbb{H}_{DB_0}^p = \mathbb{H}_D^p$ with equivalent quasi-norms. Then for any $b \in H^\infty\left(S_\mu\right)$,

$$\|b(DB_0)h\|_{H^p} \lesssim \|b\|_\infty \|h\|_{H^p}, \qquad \forall\ h \in \overline{\mathbf{R}_2(D)}.$$

Let $\frac{n}{n+1} < q < \infty$. Assume that $\mathbb{H}_{DB_0^*}^q = \mathbb{H}_D^q$ with equivalent quasi-norms. Then for any $b \in H^\infty\left(S_\mu\right)$,

$$\|\mathbb{P} b(B_0 D)h\|_{(H^q)'} \lesssim \|b\|_\infty \|\mathbb{P} h\|_{(H^q)'}, \qquad \forall\ h \in \overline{\mathbf{R}_2(B_0 D)}.$$

This result can be found in [AS14a, Theorem 4.19, Theorem 4.21].

In the above lemma the dual of H^q, for a duality extending the L^2 sesquilinear pairing, when $q>1$ is thus $H^{q'}$ and is the space $\dot{\Lambda}^{n\left(\frac{1}{q}-1\right)}$ when $q\le 1$. Here, $\dot{\Lambda}^0$ denotes BMO; for $0<\alpha<1$, $\dot{\Lambda}^\alpha$ is the Hölder space of those continuous functions with

$$|f(x)-f(y)|\le C|x-y|^s$$

which is equipped with a semi-norm.

5.3.2 Duality

We shall apply the following result for $T=DB_0$ (or DB_0^*) and $T^*=B_0^*D$ (or B_0D).

Lemma 5.3.2. *Let $\mathscr{T}=T^p$, $0<p<\infty$ and $\mathscr{T}^*$ be its dual space. Denote by $\langle,\rangle$ the L^2 sesquilinear inner product. Then for any $h\in\mathbb{H}_T^{\mathscr{T}}$, $\widetilde{h}\in\mathbb{H}_{T^*}^{\mathscr{T}^*}$*

$$\left|\langle h,\widetilde{h}\rangle\right|\lesssim\|h\|_{\mathbb{H}_T^{\mathscr{T}}}\|\widetilde{h}\|_{\mathbb{H}_{T^*}^{\mathscr{T}^*}}.$$

More generally, for any $h\in\overline{\mathbf{R}_2(T)}$, $\widetilde{h}\in\overline{\mathbf{R}_2(T^*)}$ and any $\psi,\varphi\in\Psi(S_\mu)$ for which the Calderón reproducing formula

$$\int_0^\infty\varphi(tz)\psi(tz)\frac{dt}{t}=1,\qquad\forall z\in S_\mu$$

holds, one has

$$\left|\langle h,\widetilde{h}\rangle\right|\lesssim\left\|\mathbb{Q}_{\psi,T}h\right\|_{\mathscr{T}}\left\|\mathbb{Q}_{\varphi^*,T^*}\widetilde{h}\right\|_{\mathscr{T}^*}.$$

Next, for any $g\in\mathbb{H}_{T^*}^{\mathscr{T}^*}$,

$$\|\widetilde{h}\|_{\mathbb{H}_{T^*}^{\mathscr{T}^*}}\simeq\sup\left\{\left|\langle h,\widetilde{h}\rangle\right|;f\in\mathscr{T},\|f\|_{\mathbb{H}_T^{\mathscr{T}}}=1\right\}.$$

When $1<p<\infty$, we can revert the roles of $\mathscr{T}$ and $\mathscr{T}^*$, that is, $\langle,\rangle$ is a duality pairing for the pair of spaces $\left(\mathbb{H}_T^p,\mathbb{H}_{T^*}^{p'}\right)$.

This result can be found in [AS14a, Proposition 4.8].

5.3.3 Molecular theory

For $0<p\le 1$, we can take advantage of the notion of molecules. For a cube (or a ball) $Q\subset\mathbb{R}^n$ denote the dyadic annuli by $S_i(Q)$, which is defined by $S_i(Q):=2^{i+1}Q\backslash 2^iQ$ for $i=2,3,\cdots$ and $S_1(Q):=2Q$. Here λQ is the cube with same center as Q and side-length $\lambda\ell(Q)$. Let $0<p\le 1$, $\epsilon>0$ and $M\in\mathbb{N}$. We say that a function $m\in L^2$ is a $\left(\mathbb{H}_T^p,\epsilon,M\right)$-molecule if there exists a cube $Q\subset\mathbb{R}^n$ and a function $b\in\mathbf{R}_2(T^M)$ such that $T^Mb=m$ and for $k=0,1,2,\cdots,M$

$$\left\|(\ell(Q)T)^{-k}m\right\|_{L^2(S_i(Q))}\le\left(2^i\ell(Q)\right)^{\frac{n}{2}-\frac{n}{p}}2^{-i\epsilon},\ i\in\mathbb{N}^*.\tag{5.3.3}$$

Remark that $m\in\mathbf{R}_2(T)$ and also that $m\in L^p$ with $\|m\|_p\lesssim 1$ independently of Q.

Definition 5.3.3. Let $0 < p \le 1$, $\epsilon > 0$ and $M \in \mathbb{N}^*$. For $f \in \overline{\mathbf{R}_2(T)}$, $f = \sum_j \lambda_j m_j$ is a molecular $\left(\mathbb{H}^p_T, \epsilon, M\right)$-representation of f if each m_j is an $\left(\mathbb{H}^p_T, \epsilon, M\right)$-molecule, $\{\lambda_j\} \in \ell^p$ and the series converges in L^2. We define

$$\mathbb{H}^p_{T,\mathrm{mol},M} := \left\{ f \in \overline{\mathbf{R}_2(T)}; f \text{ has a molecular } \left(\mathbb{H}^p_T, \epsilon, M\right)\text{-representation} \right\}$$

with the quasi-norm (it is a norm only when $p = 1$)

$$\|f\|_{\mathbb{H}^p_{T,\mathrm{mol},M}} := \inf \left\{ \|\{\lambda_j\}\|_{\ell^p} := \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{\frac{1}{p}} \right\},$$

taken over all molecular $\left(\mathbb{H}^p_T, \epsilon, M\right)$-representations $f = \sum_{j=0}^{\infty} \lambda_j m_j$.

Lemma 5.3.4. *Let $0 < p \le 1$, $M \in \mathbb{N}^*$ with $M > \frac{n}{p} - \frac{n}{2}$. Then*

$$\mathbb{H}^p_{T,mol,M} = \mathbb{H}^p_T$$

with equivalence of quasi-norms. In the case $T = D$, for $\frac{n}{n+1} < p \le 1$, then

$$\mathbb{H}^p_D = \mathbb{H}^p_{D,mol,1}$$

with equivalence of quasi-norms.

This result can be found in [AS14a, Proposition 4.12, Theorem 4.16].

5.3.4 Identification

Let

$$a := \inf \left\{ p \in \left(\frac{n}{n+1}, \infty \right) : \mathbb{H}^p_{DB_0} = \mathbb{H}^p_D \right\}.$$

Similarly, let

$$\tilde{a} := \inf \left\{ \tilde{p} \in \left(\frac{n}{n+1}, \infty \right) : \mathbb{H}^{\tilde{p}}_{DB_0^*} = \mathbb{H}^{\tilde{p}}_D \right\}.$$

Recall for $-1 \le \beta < 1$, we defined $p_{*,\beta} = \frac{np}{n + \frac{(1-\beta)}{2} p}$.

Lemma 5.3.5. *Let $\mathbb{P}$ be the projection as before.*

i) Let $\sigma > 0$. For $p \in \left((p_-)_*, p_+ \right)$, there holds

$$\left\| (t|DB_0|)^\sigma e^{-t|DB_0|} h \right\|_{T^p} \simeq \|h\|_{H^p}, \quad \forall\, h \in \overline{\mathbf{R}_2(DB_0)} = \overline{\mathbf{R}_2(D)}. \tag{5.3.4}$$

ii) Let $-1 \le \beta < 1$ so that $\sigma = \frac{1-\beta}{2} > 0$. Assume $\tilde{p} \in \left((\tilde{p}_-)_{*,\beta}, \tilde{p}_+ \right)$.

If $\tilde{p} > 1$ and $p = (\tilde{p})'$, there holds

$$\left\| (t|B_0 D|)^\sigma e^{-t|B_0 D|} h \right\|_{T^p} \simeq \|\mathbb{P} h\|_p, \quad \forall\, h \in \overline{\mathbf{R}_2(B_0 D)}, \tag{5.3.5}$$

and, if $\widetilde{p} \le 1$, and $\alpha = n\left(\frac{1}{\widetilde{p}} - 1\right)$, there holds

$$\left\| (t|B_0D|)^{\sigma}\, e^{-t|B_0D|} h \right\|_{T^{\infty,\alpha}} \simeq \|\mathbb{P}h\|_{\dot{\Lambda}^{\alpha}}, \;\; \forall\, h \in \overline{\mathbf{R}_2(B_0D)}. \tag{5.3.6}$$

iii) Assume $\widetilde{p} \in (\widetilde{a}, \widetilde{p}_+)$. If $\widetilde{p} > 1$ and $p = (\widetilde{p})'$, there holds

$$\left\| tDe^{-t|B_0D|} h \right\|_{T^p} \simeq \|\mathbb{P}h\|_{L^p}, \;\; \forall\, h \in \overline{\mathbf{R}_2(B_0D)}, \tag{5.3.7}$$

and, if $\widetilde{p} \le 1$, and $\alpha = n\left(\frac{1}{\widetilde{p}} - 1\right)$, there holds

$$\left\| tDe^{-t|B_0D|} h \right\|_{T^{\infty,\alpha}} \simeq \|\mathbb{P}h\|_{\dot{\Lambda}^{\alpha}}, \;\; \forall\, h \in \overline{\mathbf{R}_2(B_0D)}. \tag{5.3.8}$$

iv) For $p \in (a, p_+)$ we have

$$\left\| e^{-t|DB_0|} h \right\|_{\widetilde{T}^p} \simeq \|h\|_{H^p}, \;\; \forall\, h \in \overline{\mathbf{R}_2(DB_0)}^{\pm}, \tag{5.3.9}$$

where $\overline{\mathbf{R}_2(DB_0)}^{\pm} = \chi^{\pm}(DB_0)\overline{\mathbf{R}_2(DB_0)}$.

Proof. Part i) is proved in Theorem 5.1 of [AS14a].

Part ii) follows from Theorem 5.3 of [AS14a]. Indeed, if suffices to verify

$$\widetilde{p}_- \ge \widetilde{p} > (\widetilde{p}_-)_{*,\beta} \;\&\; \widetilde{p} > 1 \Longleftrightarrow \sigma > \frac{n}{\widetilde{p}} - \frac{n}{\widetilde{p}_-} = \frac{n}{p_+} - \frac{n}{p}$$

and

$$\widetilde{p}_- \ge \widetilde{p} > (\widetilde{p}_-)_{*,\beta} \;\&\; \widetilde{p} \le 1 \Longleftrightarrow \sigma > \frac{n}{\widetilde{p}} - \frac{n}{\widetilde{p}_-} = \alpha + \frac{n}{p_+}.$$

Note that there is a typo in Theorem 5.3 of [AS14a] where $\frac{n}{p} - \frac{n}{p_+}$ should read as $\frac{n}{p_+} - \frac{n}{p}$.

Part iii) is proved in Theorem 5.8 of [AS14a].

Part iv) is proved in Theorem 9.1 in [AS14a]. □

5.4 Duality in trace spaces

This section is devoted to the study of duality in boundary trace spaces. We rely heavily on Lemma 5.3.5. We also study the relation of this duality with the condition $\mathbf{E} \in \widetilde{T}^{\infty}$. We first look at the duality results for intermediate weights $\beta \in (-1, 1)$.

Lemma 5.4.1. *Let $\beta \in (-1,1)$ and $\mathbf{E} \in L^{\infty}$. Let $\widetilde{\mathbf{E}} = B_0^{-1}\mathbb{P}_{B_0D}\mathbf{E}$*[2].

I) For $p \in (p_-, p_+)$, the operator

$$\mathbf{I}^{\pm}_{\beta}(\widetilde{\mathbf{E}}\cdot) : L^2_c(\mathbb{R}^{1+n}_+; \mathbb{C}^N) \ni F \mapsto \int_0^{\infty} |DB_0|^{\frac{1-\beta}{2}} e^{-s|DB_0|} \chi^{\pm}(DB_0) \left(\widetilde{\mathbf{E}}F\right)_s ds,$$

2. $\mathbf{E}$ is viewed as a multiplication while $\widetilde{\mathbf{E}}$ is only an abstract L^2 defined operator.

satisfies the bound

$$\left\| \mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right)\right\|_{\mathbb{H}^{p}_{DB_0}} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{T^p_\beta},$$

thereby,

$$\left\| |DB_0|^{\frac{1+\beta}{2}} e^{-t|DB_0|}\mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right)\right\|_{T^p_\beta} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{T^p_\beta}. \tag{5.4.1}$$

II) For $\widetilde{p} \in \left(\left(\widetilde{p}_-\right)_{*,-\beta}, \widetilde{p}_+\right)$, *the operator*

$$\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}\cdot) : L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right) \ni F \mapsto \int_0^\infty |B_0D|^{\frac{1-\beta}{2}} e^{-s|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\, ds,$$

satisfies the bound

$$\left\| \widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'},$$

thereby,

$$\left\| |B_0D|^{\frac{1+\beta}{2}} e^{-t|B_0D|}\,\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F)\right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'} \lesssim \|\mathbf{E}\|_{\infty}\|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}. \tag{5.4.2}$$

We will not use part II) of the above lemma in estimating $\mathbf{S}_{DB_0}$, but it can be used to estimate $\mathbf{M}_{B_0D} = B_0\mathbf{S}_{DB_0}$. This will not be presented in this thesis.

We do not know how to show similar trace space duality results as in part I) of Lemma 5.4.1 outside the functional calculus interval without assuming $B_0^{-1} \in L^\infty$. We give, however, the following variant of Lemma 5.4.1.

Lemma 5.4.2. *Let* $\beta \in (-1,1)$ *and* $\mathbf{E} \in L^\infty$. *Assume* $B_0^{-1} \in L^\infty$ *so that* $\widetilde{\mathbf{E}} = B_0^{-1}\mathbf{E} \in L^\infty$.

I) For $1 < p \in \left(\left(p_-\right)_{*,\beta}, p_+\right)$, *the operator*

$$\mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}\cdot\right) : L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right) \ni F \mapsto \int_0^\infty |DB_0|^{\frac{1-\beta}{2}} e^{-s|DB_0|}\chi^{\pm}(DB_0)\left(\widetilde{\mathbf{E}}F\right)_s ds,$$

satisfies the bound

$$\left\| \mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right)\right\|_{\mathbb{H}^{p}_{DB_0}} \lesssim \|\widetilde{\mathbf{E}}\|_{\infty}\|F\|_{T^p_\beta},$$

thereby,

$$\left\| |DB_0|^{\frac{1+\beta}{2}} e^{-t|DB_0|}\mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right)\right\|_{T^p_\beta} \lesssim \|\widetilde{\mathbf{E}}\|_{\infty}\|F\|_{T^p_\beta}. \tag{5.4.3}$$

II) For $\widetilde{p} \in \left(\left(\widetilde{p}_-\right)_{*,-\beta}, \widetilde{p}_+\right)$, *one has*

$$\left\| B_0^{-1}|B_0D|^{\frac{1+\beta}{2}} e^{-t|B_0D|}\,\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F)\right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'} \lesssim \|\widetilde{\mathbf{E}}\|_{\infty}\|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}. \tag{5.4.4}$$

Here $\widetilde{\mathbf{I}}^{\pm}_{\beta}$ *are defined in Lemma 5.4.1.*

Note that for part II) in the above lemma, unlike its counterpart in Lemma 5.4.1, we can write formally: for $B_0^{-1} \in L^\infty$ and $\mathbf{E}F \in L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N)$

$$B_0^{-1}|B_0D|^{\frac{1+\beta}{2}} e^{-t|B_0D|}\,\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F)_t = \int_0^\infty De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s ds,$$

which is also equal to $|DB_0|^{\frac{1+\beta}{2}} e^{-t|DB_0|}\mathbf{I}^{\pm}_{\beta}(\widetilde{\mathbf{E}}F)_t$.

The role of the Carleson-Dahlberg function $\mathbf{E} \in \widetilde{T}^\infty$ is seen in the following lemma which treats the case of endpoint weights for Lemma 5.4.2. Recall from [HR13] and Chapter 4 that

$$\|\mathbf{E}F\|_{T^p} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p}, \quad \forall\, 0<p<\infty.$$

Lemma 5.4.3. *Assume* $\mathbf{E} \in \widetilde{T}^\infty$.

I) For p such that $\mathbb{H}^p_{DB_0} = \mathbb{H}^p_D$, hence for $p \in ((p_-)_*, p_+)$, the operator

$$\mathbf{U}^{\pm}(\mathbf{E}\cdot) : L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N) \ni F \mapsto \int_0^\infty e^{-s|DB_0|}\chi^{\pm}(DB_0)D(\mathbf{E}F)_s ds,$$

satisfies the bound

$$\left\|\mathbf{U}^{\pm}(\mathbf{E}F)\right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p},$$

thereby,

$$\left\|e^{-t|DB_0|}\mathbf{U}^{\pm}(\mathbf{E}F)\right\|_{\widetilde{T}^p} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p}. \tag{5.4.5}$$

II) For $\widetilde{p}$ such that $\mathbb{H}^{\widetilde{p}}_{DB_0^*} = \mathbb{H}^{\widetilde{p}}_D$, hence for $\widetilde{p} \in ((\widetilde{p}_-)_*, \widetilde{p}_+)$, the operator

$$\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}\cdot) : L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N) \ni F \mapsto \int_0^\infty e^{-s|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s ds,$$

satisfies the bound

$$\left\|\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{(T^{\widetilde{p}})'},$$

thereby,

$$\left\|De^{-t|B_0D|}\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{(T^{\widetilde{p}})'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{(T^{\widetilde{p}})'}. \tag{5.4.6}$$

We see that part I) and part II) of the above lemma correspond respectively to $\beta = -1$ in I) and $\beta = 1$ in II) of Lemma 5.4.2.

5.4.1 Proof of Lemma 5.4.1

For the proof of part I), we need a little bit of the Sobolev theory for perturbed first order Dirac operators. For $1<p<\infty$, let $\dot{\mathbb{W}}^{-1,p}_{DB_0}$ be the space of functions $h \in \overline{\mathbf{R}_2(DB_0)} =$

$\mathbb{H}^2_{DB_0}$ such that $\left\|\tau\mathbb{Q}_{\psi,DB_0}h\right\|_{T^p}<\infty$ for any choice of ψ among bounded holomorphic functions in bisectors S_μ with enough decay at 0 and ∞. This space does not depend on the choice of ψ as above. We let $\dot{W}^{-1,p}_{DB_0}$ be its completion.

Also for $1<q<\infty$, let $\dot{\mathbb{W}}^{1,q}_{B_0^*D}$ be the space of functions $h\in\overline{\mathbf{R}_2(B_0^*D)}=\mathbb{H}^2_{B_0^*D}$ with $\tau^{-1}\mathbb{Q}_{\psi,B_0^*D}h\in T^q$, equipped with the norm $\left\|\tau^{-1}\mathbb{Q}_{\psi,B_0^*D}h\right\|_{T^q}$. Again, this space does not depend on the particular choice of ψ as above. Let $\dot{W}^{1,q}_{B_0^*D}$ be its completion.

When $q=p'$, the two spaces $\dot{\mathbb{W}}^{1,q}_{B_0^*D}$ and $\dot{\mathbb{W}}^{-1,p}_{DB_0}$ are in duality for the L^2 duality. It extends to completion. For q such that $\mathbb{H}^q_{DB_0^*}=\mathbb{H}^q_D$ and $p'=q$, we have that $\dot{W}^{-1,p}_{DB_0}=\dot{W}^{-1,p}_D$ with equivalence of norms and $\mathbb{P}$ extends to an isomorphism from $\dot{W}^{1,q}_{B_0^*D}$ onto $\dot{W}^{1,q}_D$. See [AM14, AS14a].

Proof of part I) of Lemma 5.4.1. Recall that by [AS14a, Proposition 4.17] we have

$$\|h\|_{\mathbb{H}^p_D}\lesssim\left\|\mathbb{Q}_{\psi,DB_0}h\right\|_{T^p},\qquad p<2\text{ and }\psi\in\Psi^{\gamma(p)}.$$

Using this property, for $\sigma>0$ and $\psi(z)=z^{\sigma+1}e^{-z}$ we can reduce the desired Hardy space estimate to a T^p estimate for the following mapping

$$L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)\ni F\mapsto\int_0^\infty(t|DB_0|)^{\sigma+1}e^{-t|DB_0|}|DB_0|^{\frac{1-\beta}{2}}e^{-s|DB_0|}\chi^\pm(DB_0)\left(\widetilde{\mathbf{E}}F\right)_s ds$$
$$=\int_0^\infty t(t|DB_0|)^\sigma e^{-t|DB_0|}|DB_0|^{\frac{1-\beta}{2}}e^{-s|DB_0|}\chi^\pm(DB_0)D\mathbb{P}_{B_0D}(\mathbf{E}F)_s ds.$$

Let

$$G=\int_0^\infty|DB_0|^{\frac{1-\beta}{2}}e^{-s|DB_0|}\chi^\pm(DB_0)D\mathbb{P}_{B_0D}(\mathbf{E}F)_s ds.$$

Choosing σ large enough, this further reduces to estimate G in $\dot{\mathbb{W}}^{-1,p}_{DB_0}$.

Now, we have

$$\|G\|_{\dot{\mathbb{W}}^{-1,p}_{DB_0}}\simeq\sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1}\left|\int_0^\infty\left\langle\mathbb{P}_{B_0D}s^{\frac{1+\beta}{2}}(\mathbf{E}F)_s,D\left(s|B_0^*D|\right)^{\frac{1-\beta}{2}}e^{-s|B_0^*D|}\chi^\pm(B_0^*D)h\right\rangle\frac{ds}{s}\right|$$
$$\simeq\sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1}\left|\int_0^\infty\left\langle s^{\frac{1+\beta}{2}}(\mathbf{E}F)_s,D\left(s|B_0^*D|\right)^{\frac{1-\beta}{2}}e^{-s|B_0^*D|}\chi^\pm(B_0^*D)h\right\rangle\frac{ds}{s}\right|$$
$$\lesssim\sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1}\left\|\left(s|DB_0^*|\right)^{\frac{1-\beta}{2}}e^{-s|DB_0^*|}\chi^\pm(DB_0^*)Dh\right\|_{T^{p'}}\left\|s^{\frac{\beta-1}{2}}(\mathbf{E}F)(s,\cdot)\right\|_{T^p_{+1}}$$

$$
\begin{aligned}
&\lesssim \sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1} \|Dh\|_{p'} \left\| s^{\frac{\beta-1}{2}} (\mathbf{E}F)(s,\cdot)\right\|_{T^p_{+1}} \\
&\lesssim \sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1} \|\mathbb{P}h\|_{\dot{W}^{1,p'}} \left\| s^{\frac{\beta-1}{2}} (\mathbf{E}F)(s,\cdot)\right\|_{T^p_{+1}} \\
&\lesssim \sup_{\|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}}=1} \|h\|_{\dot{\mathbb{W}}^{1,p'}_{B_0^*D}} \left\| s^{\frac{\beta-1}{2}} (\mathbf{E}F)(s,\cdot)\right\|_{T^p_{+1}} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_\beta}.
\end{aligned}
$$

We used (5.2.1) in removing $\mathbb{P}_{B_0D}$. Note that $T^p_{+1} = \left(T^{p'}\right)'$. We used Lemma 5.3.5 (for DB_0^*) in the second inequality. Note that in both the second and the fourth inequality we need the requirement that $\mathbb{H}^{p'}_{DB_0^*} = \mathbb{H}^{p'}_D$.

Thereby, since $\frac{1+\beta}{2} > 0$ (as $\beta > -1$) we have

$$
\begin{aligned}
&\left\| |DB_0|^{\frac{1+\beta}{2}} e^{-t|DB_0|} \mathbf{I}^\pm_\beta \left(\widetilde{\mathbf{E}}F\right)\right\|_{T^p_\beta} \\
&\qquad \simeq \left\| \left(t|DB_0|\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|} \mathbf{I}^\pm_\beta \left(\widetilde{\mathbf{E}}F\right)\right\|_{T^p} \\
&\qquad \lesssim \left\| \mathbf{I}^\pm_\beta \left(\widetilde{\mathbf{E}}F\right)\right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_\beta}.
\end{aligned}
$$

We used Lemma 5.3.5 in the first inequality. This requires $\mathbb{H}^p_{DB_0} = \mathbb{H}^p_D$.

The intersection of the above two requirements on p forces $p \in (p_-, p_+)$, which meets our assumption on p. $\square$

Proof of part II) of Lemma 5.4.1. The proof is a consequence of duality arguments, and it covers the cases $\widetilde{p} > 1$ and $\widetilde{p} \le 1$ at the same time. Indeed, we have

$$
\begin{aligned}
\left\|\widetilde{\mathbf{I}}^\pm_\beta (\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'} &\simeq \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \left| \int_0^\infty \left\langle s^{\frac{1+\beta}{2}} (\mathbf{E}F)_s, (s|DB_0^*|)^{\frac{1-\beta}{2}} e^{-s|DB_0^*|} \chi^\pm(DB_0^*) h \right\rangle \frac{ds}{s} \right| \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \left\| (s|DB_0^*|)^{\frac{1-\beta}{2}} e^{-s|DB_0^*|} \chi^\pm(DB_0^*) h \right\|_{T^{\widetilde{p}}} \left\| s^{\frac{-1+\beta}{2}} (\mathbf{E}F)(s,\cdot)\right\|_{(T^{\widetilde{p}})'} \\
&\lesssim \|\mathbf{E}\|_\infty \|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}.
\end{aligned}
$$

We used Lemma 5.3.5 for DB_0^* for $\widetilde{p} \in \left((\widetilde{p}_-)_*, \widetilde{p}_+\right)$, since $\frac{1-\beta}{2} > 0$ (as $\beta < 1$). We also used that $s^{\frac{-1+\beta}{2}} (\mathbf{E}F)(s,\cdot) \in \left(T^{\widetilde{p}}\right)'$ is equivalent to $(\mathbf{E}F) \in \left(T^{\widetilde{p}}_{-\beta}\right)'$.

Thereby, since $\widetilde{p}$ is also in $\left((\widetilde{p}_-)_{*,-\beta}, \widetilde{p}_+\right)$, we have

$$
\left\| |B_0D|^{\frac{1+\beta}{2}} e^{-t|B_0D|} \widetilde{\mathbf{I}}^\pm_\beta (\mathbf{E}F)\right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}
$$

$$
\begin{aligned}
&= \left\| (t|B_0D|)^{\frac{1+\beta}{2}} e^{-t|B_0D|}\, \widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F) \right\|_{\left(T^{\tilde p}_{+1}\right)'} \\
&\lesssim \left\| \mathbb{P}\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F) \right\|_{(H^{\tilde p})'} \\
&\simeq \left\| \mathbb{P}\widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F) \right\|_{\left(\mathbb{H}^{\tilde p}_{D}\right)'} \lesssim \left\| \widetilde{\mathbf{I}}^{\pm}_{\beta}(\mathbf{E}F) \right\|_{\left(\mathbb{H}^{\tilde p}_{DB_0^*}\right)'} \lesssim \|\mathbf{E}\|_\infty \|F\|_{\left(T^{\tilde p}_{-\beta}\right)'}.
\end{aligned}
$$

We used Lemma 5.3.5 and Lemma 5.3.1 in order in the first two inequalities. □

5.4.2 Proof of Lemma 5.4.2

The claim in part II) follows from part II) of Lemma 5.4.1.

Proof of part I) in Lemma 5.4.2. The proof is a consequence of duality arguments, so it is limited to the case $p > 1$. Note that we have

$$
\begin{aligned}
\left\| \mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right) \right\|_{\mathbb{H}^p_{DB_0}} &\simeq \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left| \int_0^\infty \left\langle s^{\frac{1+\beta}{2}} \left(\widetilde{\mathbf{E}}F\right)_s, (s|B_0^*D|)^{\frac{1-\beta}{2}} e^{-s|B_0^*D|} \chi^{\pm}(B_0^*D)h \right\rangle \frac{ds}{s} \right| \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left\| (s|B_0^*D|)^{\frac{1-\beta}{2}} e^{-s|B_0^*D|} \chi^{\pm}(B_0^*D)h \right\|_{T^{p'}} \left\| s^{\frac{\beta-1}{2}} \left(\widetilde{\mathbf{E}}F\right)(s,\cdot) \right\|_{T^p_{+1}} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \|\mathbb{P}h\|_{\mathbb{H}^{p'}_{D}} \left\| s^{\frac{\beta-1}{2}} \left(\widetilde{\mathbf{E}}F\right)(s,\cdot) \right\|_{T^p_{+1}} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \|h\|_{\mathbb{H}^{p'}_{B_0^*D}} \left\| s^{\frac{\beta-1}{2}} \left(\widetilde{\mathbf{E}}F\right)(s,\cdot) \right\|_{T^p_{+1}} \lesssim \|\widetilde{\mathbf{E}}\|_\infty \|F\|_{T^p_\beta}.
\end{aligned}
$$

Note that $T^p_{+1} = (T^p)'$. In the last two lines we used Lemma 5.3.5 and Lemma 5.3.1 (precisely their analogues stated for B_0^*D instead of B_0D) and $p \in \left((p_-)_{*,\beta}, p_+\right)$.

Thereby, since p is also in $\left((p_-)_*, p_+\right)$ (as $\beta > -1$), we have

$$
\begin{aligned}
&\left\| |DB_0|^{\frac{1+\beta}{2}} e^{-t|DB_0|} \mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right) \right\|_{T^p_\beta} \\
&\qquad \simeq \left\| (t|DB_0|)^{\frac{1+\beta}{2}} e^{-t|DB_0|} \mathbf{I}^{\pm}_{\beta}\left(\widetilde{\mathbf{E}}F\right) \right\|_{T^p} \\
&\qquad \lesssim \left\| \mathbf{I}^{\pm}_{\beta}(\widetilde{\mathbf{E}}F) \right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\widetilde{\mathbf{E}}\|_\infty \|F\|_{T^p_\beta}.
\end{aligned}
$$

We used Lemma 5.3.5 in the first inequality. □

5.4.3 Proof of Lemma 5.4.3

Recall that for $\frac{n}{n+1} < p \le 1$, we have $\mathbb{H}^p_D = \mathbb{H}^p_{D,\mathrm{mol},1}$ by Lemma 5.3.4.

Proof of Lemma 5.4.3. We prove the claim in II) first. The arguments cover the cases $\widetilde{p} > 1$ and $\widetilde{p} \le 1$ at the same time. Indeed, we have

$$\begin{aligned}
\left\|\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'} &\simeq \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \left|\int_0^\infty \left\langle (\mathbf{E}F)_s, e^{-s|DB_0^*|}\chi^{\pm}(DB_0^*)h\right\rangle ds\right| \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \left\|\overline{\mathbf{E}}\cdot e^{-s|DB_0^*|}\chi^{\pm}(DB_0^*)h\right\|_{T^{\widetilde{p}}} \|F\|_{(T^{\widetilde{p}})'} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \|\mathbf{E}\|_{\widetilde{T}^\infty} \left\|e^{-s|DB_0^*|}\chi^{\pm}(DB_0^*)h\right\|_{\widetilde{T}^{\widetilde{p}}} \|F\|_{(T^{\widetilde{p}})'} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \|\mathbf{E}\|_{\widetilde{T}^\infty} \left\|\chi^{\pm}(DB_0^*)h\right\|_{\mathbb{H}^{\widetilde{p}}_D} \|F\|_{(T^{\widetilde{p}})'} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{\widetilde{p}}_{DB_0^*}}=1} \|\mathbf{E}\|_{\widetilde{T}^\infty} \|h\|_{\mathbb{H}^{\widetilde{p}}_D} \|F\|_{(T^{\widetilde{p}})'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty} \|F\|_{(T^{\widetilde{p}})'}.
\end{aligned}$$

Here $\overline{\mathbf{E}}(t,y) = \overline{\mathbf{E}(t,y)}$ when $(t,y) \in \mathbb{R}^{1+n}_+$, hence $\|\mathbf{E}\|_{\widetilde{T}^\infty} = \|\overline{\mathbf{E}}\|_{\widetilde{T}^\infty}$. In the second inequality we used the multiplication of tent spaces from [HR13] and Chapter 4. In the third inequality we used Lemma 5.3.5 (for DB_0^*). We used respectively Lemma 5.3.1 (for DB_0^*) and the assumption $\mathbb{H}^{\widetilde{p}}_{DB_0^*} = \mathbb{H}^{\widetilde{p}}_D$ in the last two steps.

Thereby, we have the following estimates

$$\begin{aligned}
&\left\|De^{-t|B_0D|}\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{(T^{\widetilde{p}})'} \\
&\quad\lesssim \left\|\mathbb{P}\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{(H^{\widetilde{p}})'} \\
&\quad\simeq \left\|\mathbb{P}\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_D\right)'} \\
&\quad\lesssim \left\|\widetilde{\mathbf{U}}^{\pm}(\mathbf{E}F)\right\|_{\left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{(T^{\widetilde{p}})'}.
\end{aligned}$$

We used Lemma 5.3.5 and Lemma 5.3.1 in the first two inequalities.

For the claim in I) in the reflexive case, namely

$$\left\|\mathbf{U}^{\pm}(\mathbf{E}F)\right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\mathbf{E}F\|_{T^p}, \quad p > 1,$$

the arguments are similar to the proof of the claim in part II). Indeed, we have

$$\begin{aligned}
\left\|\mathbf{U}^{\pm}(\mathbf{E}F)\right\|_{\mathbb{H}^p_{DB_0}} &\simeq \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left|\int_0^\infty \left\langle (\mathbf{E}F)_s, De^{-s|B_0^*D|}\chi^{\pm}(B_0^*D)h\right\rangle ds\right| \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left\|De^{-s|B_0^*D|}\chi^{\pm}(B_0^*D)h\right\|_{(T^p)'} \|\mathbf{E}F\|_{T^p}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left\| De^{-s|B_0^*D|}\chi^{\pm}(B_0^*D)h \right\|_{(T^p)'} \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \left\|\mathbb{P}\chi^{\pm}(B_0^*D)h\right\|_{\mathbb{H}^{p'}_D} \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p} \\
&\lesssim \sup_{\|h\|_{\mathbb{H}^{p'}_{B_0^*D}}=1} \|\mathbb{P}h\|_{\mathbb{H}^{p'}_D}\|F\|_{\widetilde{T}^p} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty}\|F\|_{\widetilde{T}^p}.
\end{aligned}$$

In the second inequality we used the multiplication of tent spaces from [HR13] and Chapter 4. In the third inequality we used Lemma 5.3.5 (for B_0^*D). We used Lemma 5.3.1 (for B_0^*D) in the last two steps.

For the claim in I) in the non-reflexive case, namely

$$\left\|\mathbf{U}^{\pm}(\mathbf{E}F)\right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\mathbf{E}F\|_{T^p}, \quad p \le 1,$$

we can use the atomic decomposition of tent spaces and the molecular decomposition of Hardy spaces $\mathbb{H}^p_{DB_0}$. Write

$$\mathbf{U}^{\pm}(\mathbf{E}F) = \chi^{\pm}(DB_0)\widehat{\mathbf{U}}^{\pm}(\mathbf{E}F)$$

where

$$\widehat{\mathbf{U}}^{\pm}(\mathbf{E}F)_t := \int_0^\infty sD\phi^{\pm}(sB_0D)(\mathbf{E}F)_s \frac{ds}{s},$$

where $\phi^{\pm}$ are the approximation families obtained via Lemma 5.2.10 such that $\left\{sD\phi^{\pm}(sB_0D)\right\}_{s>0}$ has large enough $L^2 - L^2$ off-diagonal decay. It suffices to bound

$$m_{\pm} = \int_0^{\ell(Q)} sD\phi^{\pm}(sB_0D)A_s \frac{ds}{s}$$

where A is a T^p-atom supported on $T(Q)$ for some cube $Q \subset \mathbb{R}^n$. Since $\phi^{\pm}(sB_0D)$ and $sD\phi^{\pm}(sB_0D)$ have large order $L^2 - L^2$ off-diagonal decay, it can be verified that $m_{\pm}$ is an $\left(\mathbb{H}^p_D, \varepsilon, 1\right)$-molecule (for any $\varepsilon > 0$) associated to Q. Indeed, by letting

$$b_{\pm} = \int_0^{\ell(Q)} \phi^{\pm}(sB_0D)A_s ds,$$

we see that $m_{\pm} = Db_{\pm} \in L^2$ and it suffices to check that for $k = 0, 1$

$$\left\|(\ell(Q)D)^{-k} m\right\|_{L^2(S_i(Q))} \le \left(2^i\ell(Q)\right)^{\frac{n}{2}-\frac{n}{p}} 2^{-i\epsilon}, \; i \in \mathbb{N}^*. \tag{5.4.7}$$

The case for $k = 0$ is a direct consequence of $L^2 - L^2$ off-diagonal decay and the size requirement on T^p-atoms. The case for $k = 1$ needs to use Cauchy-Schwarz inequality in the s-integral in $b_{\pm}$ first, that is,

$$\left\| \frac{1}{\ell(Q)} \int_0^{\ell(Q)} \phi^{\pm}(sB_0D)A_s ds \right\|_{L^2(S_i(Q))}$$

$$\leq \frac{1}{\ell(Q)} \int_0^{\ell(Q)} \left\| \phi^{\pm}(sB_0D)A_s \right\|_{L^2(S_i(Q))} ds$$
$$\leq \frac{1}{\ell(Q)} \left(\int_0^{\ell(Q)} s^2 \frac{ds}{s} \right)^{1/2} \left(\int_0^{\ell(Q)} \left\| \phi^{\pm}(sB_0D)A_s \right\|^2_{L^2(S_i(Q))} \frac{ds}{s} \right)^{1/2}$$
$$\leq \left(\int_0^{\ell(Q)} \left\| \phi^{\pm}(sB_0D)A_s \right\|^2_{L^2(S_i(Q))} \frac{ds}{s} \right)^{1/2}.$$

The remaining arguments in using the L^2-L^2 off-diagonal decay are similar to the case $k=0$. We omit the details.

Thereby, we have the following estimates

$$\begin{aligned} &\left\| e^{-t|DB_0|}\mathbf{U}^{\pm}(\mathbf{E}F) \right\|_{\widetilde{T}^p} \\ &\qquad \lesssim \left\| \mathbf{U}^{\pm}(\mathbf{E}F) \right\|_{\mathbb{H}^p_D} \\ &\qquad \simeq \left\| \mathbf{U}^{\pm}(\mathbf{E}F) \right\|_{\mathbb{H}^p_{DB_0}} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty} \|F\|_{\widetilde{T}^p}.. \end{aligned}$$

We used respectively Lemma 5.3.5 and the assumption $\mathbb{H}^p_{DB_0} = \mathbb{H}^p_D$. □

5.5 Proof of Theorem 5.1.1: stability

In this section we prove part 1), and part 2) when $p \in (p_-, p_+)$ and $\widetilde{p} \in (\widetilde{p}_-, \widetilde{p}_+)$, of Theorem 5.1.1 on weighted conical maximal regularity estimates of $\mathbf{S}_{DB_0}$ [3].

The complete proof of Theorem 5.1.1 is quite long. We shall break it into five steps/subsections. We present the basic manipulations in the first two subsections. These manipulations will also be useful in other sections. Meanwhile, some of the lemmata below (namely Lemma 5.5.1 and Lemma 5.5.2) are also proved in largest generality for their potential use in later sections.

Fix $\ell \in \mathbb{N}^*$ with $\ell > 2n+2$. We want to study the operator $\mathbf{S}_{DB_0}$ as defined in (5.1.3).

For $\mathbf{S}^+_{DB_0}$, we write

$$\begin{aligned} \mathbf{S}^+_{DB_0}(\mathbf{E}F)_t &= \int_0^t De^{-(t-s)|B_0D|} \left(I - e^{-2s|B_0D|}\right)^{\ell} \chi^+(B_0D)(\mathbf{E}F)_s \, ds \\ &\quad + \sum_{j=1}^{\ell} \alpha_{j\ell} \int_0^t De^{-(t+(2j-1)s)|B_0D|} \chi^+(B_0D)(\mathbf{E}F)_s \, ds \qquad (5.5.1) \\ &=: \mathbf{S}^{+,\ell}_{DB_0}(\mathbf{E}F)_t + \sum_{j=1}^{\ell} \alpha_{j\ell} \mathbf{V}^{+,+,j}_{DB_0}(\mathbf{E}F)_t, \end{aligned}$$

3. Recall that by $B_0 \in L^\infty$, this also gives the corresponding weighted conical maximal regularity estimates of $\mathbf{M}_{B_0D} = B_0\mathbf{S}_{DB_0}$.

when $\mathbf{E}F \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. Here $\alpha_{j\ell}$ are binomial coefficients. The first sign "+" in $\mathbf{V}^{+,+,j}_{DB_0}$ designates the forward integral $\int\limits_0^t$ while the second sign "+" in $\mathbf{V}^{+,+,j}_{DB_0}$ designates the projection $\chi^+(B_0D)$. Call $\mathbf{S}^{+,\ell}_{DB_0}$ the singular part of $\mathbf{S}^+_{DB_0}$ and call $\mathbf{V}^{+,+,j}_{DB_0}$ the regular parts of $\mathbf{S}^+_{DB_0}$. For $\mathbf{S}^-_{DB_0}$, we write

$$\begin{aligned}\mathbf{S}^-_{DB_0}(\mathbf{E}F)_t &= \left(I - e^{-2t|DB_0|}\right)^\ell \int\limits_t^\infty De^{-(s-t)|B_0D|}\chi^-(B_0D)(\mathbf{E}F)_s\,ds \\ &\quad + \sum_{j=1}^{\ell} \alpha_{j\ell} \int\limits_t^\infty De^{-(s+(2j-1)t)|B_0D|}\chi^-(B_0D)(\mathbf{E}F)_s\,ds \\ &=: \mathbf{S}^{-,\ell}_{DB_0}(\mathbf{E}F)_t + \sum_{j=1}^{\ell} \alpha_{j\ell}\mathbf{V}^{-,-,j}_{DB_0}(\mathbf{E}F)_t,\end{aligned} \tag{5.5.2}$$

when $\mathbf{E}F \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. The first sign "-" in $\mathbf{V}^{+,+,j}_{DB_0}$ designates the backward integral $\int\limits_t^\infty$ while the second sign "-" in $\mathbf{V}^{+,+,j}_{DB_0}$ designates the projection $\chi^-(B_0D)$. Call $\mathbf{S}^{-,\ell}_{DB_0}$ the singular part of $\mathbf{S}^-_{DB_0}$ and call $\mathbf{V}^{-,-,j}_{DB_0}$ the regular parts of $\mathbf{S}^-_{DB_0}$.

Note that the same splittings apply to $\mathbf{M}_T$ for $T \in \{DB_0, B_0D\}$. We will refer to these decompositions as *singular-regular decompositions.*

Let $\mathbf{V}^{+,+}_{DB_0} = \mathbf{V}^{+,+,1}_{DB_0}$ and $\mathbf{V}^{-,-}_{DB_0} = \mathbf{V}^{-,-,1}_{DB_0}$.

5.5.1 Singular parts

Consider

$$\mathbf{S}^{+,\ell}_{DB_0}(G)_t = \int\limits_0^t De^{-(t-s)|B_0D|}\left(I - e^{-2s|B_0D|}\right)^\ell \chi^+(B_0D)G_s\,ds,$$

and

$$\mathbf{S}^{-,\ell}_{DB_0}(G)_t = \int\limits_0^t De^{-(s-t)|B_0D|}\left(I - e^{-2t|B_0D|}\right)^\ell \chi^+(B_0D)G_s\,ds$$

when $G \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$.

The proofs of next two lemmas are given in the appendix (Section 5.8).

Lemma 5.5.1. *Fix $\ell \in \mathbb{N}^*$ with $\ell \geq n+1$. For $\beta \leq 1$, $\mathbf{S}^{+,\ell}_{DB_0}$ extends to a bounded operator on T^p_β for $\frac{n}{n+1} < p < \infty$ and is bounded on $T^{\infty,\alpha}_\beta$ for $0 \leq \alpha < 1$.*

Lemma 5.5.2. *Fix $\ell \in \mathbb{N}^*$ with $\ell \geq n+1$. For $\beta \geq -1$, $\mathbf{S}^{-,\ell}_{DB_0}$ extends to a bounded operator on T^p_β for $\frac{n}{n+1} < p < \infty$ and is bounded on $T^{\infty,\alpha}_\beta$ for $0 \leq \alpha < 1$.*

Note that the first lemma includes the case $\beta = 1$ while the second includes the case $\beta = -1$. This will be important in application.

5.5.2 Regular parts

Let $1 < p < \infty$ and let $V^p := L^p_x\left(L^2\left(\mathbb{R}_+, t^{-1}dt\right)\right)$. Given an operator-valued kernel $\{K(t,s)\}_{0<s\neq t<\infty} \subset \mathscr{L}\left(L^p(\mathbb{R}^n)\right)$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, consider

$$K_z^+(t,s) = \left[\mathbf{1}_{\mathbb{R}_+}(t-s)\right]\left(\frac{s}{t}\right)^z K(t,s),$$

and

$$K_z^-(t,s) = \left[\mathbf{1}_{\mathbb{R}_+}(s-t)\right]\left(\frac{t}{s}\right)^z K(t,s).$$

Let $K_z = K_z^+ + K_z^-$.

We have the following R-boundedness results.

Lemma 5.5.3. *Suppose $\{K(t,s)\}_{0<s\neq t<\infty}$ is an operator-valued family in $\mathscr{L}\left(L^p(\mathbb{R}^n)\right)$ which is also R-bounded in L^p, $1 < p < \infty$. For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, consider*

$$T_{K_z}(F)_t = \int_0^\infty K_z(t,s) F_s \frac{ds}{s},$$

defined[4] *for $F \in V^p$, $1 < p < \infty$. Then*

$$\| T_{K_z} \|_{V^p \to V^p} \lesssim e^{|\operatorname{Im} z|},$$

where the implicit constant may depend on $\operatorname{Re} z$, but not on $\operatorname{Im} z$.

This lemma is proved in [FMP14, Lemma 10.3] when $z = \varepsilon > 0$. The extension to $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and the dependence of the operator norm on $\operatorname{Re} z$ are straightforward. We call estimates in Lemma 5.5.3 Schur type estimates.

We now introduce specific classes in $H^\infty(S_\mu)$. Let $M \in \mathbb{N}^*$ such that $M - 1 > n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)$. We let $\mathscr{R}^M(S_\mu)$ denote the subclass of $H^\infty(S_\mu)$ of those ϕ of the form

$$\phi(z) = \sum_{k=1}^{L} c_k (1 + ikz)^{-M} \tag{5.5.3}$$

for some integer $L \geq 1$ and $c_k \in \mathbb{C}$.

Lemma 5.5.4. *Let $\phi \in \mathscr{R}^M(S_\mu)$. Then, with $T \in \{DB_0, B_0D\}$,*

(i) for $p_- < p \leq 2$

$$\left\| tT\phi(tT)F \right\|_{T^p} \lesssim \|F\|_{V^p}; \tag{5.5.4}$$

(ii) for $2 \leq p < p_+$

$$\left\| tT\phi(tT)F \right\|_{V^p} \lesssim \|F\|_{T^p}. \tag{5.5.5}$$

The same statements hold with $tT\phi(tT)$ replaced by either $tD\phi(tB_0D)$ where $\phi \in \mathscr{R}^M(S_\mu)$, or with $tT\phi(tT)$ replaced by $\psi(tT)$ where $\psi \in \Psi_\sigma^\tau$ such that $\tau > n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)$ and σ large enough.

4. This is well defined as seen from the conclusion of Lemma 5.5.3.

Proof. For the claims on $tT\phi(tT)$, it is enough to prove the one for $\phi(tT) = (I + itT)^{-M}$. Note that there is nothing to prove when $p = 2$, since V^2 is the tent space T^2 by Fubini's theorem, and we have L^2 functional calculus for the operator T.

Let $p_- < p < 2$ and find an $\varepsilon > 0$ with $p - \varepsilon > p_-$. The proof of (5.5.5) is the same as that of Lemma 3.2.3 and Lemma 3.5.1 in Chapter 3, since we have under Lemma 5.2.8, that the operator family $tT(I + itT)^{-M}$ has $L^{p-\varepsilon} - L^2$ off-diagonal decay with arbitrarily large order. The $p > 2$ case follows by dualizing with the case $p < 2$ for T^*.

The above arguments, in particular those used in the proof of Lemma 3.2.3 in Chapter 3, only make use of the large order $L^q - L^r$ off-diagonal decay of the operator family $tT\phi(tT)$ for $\phi \in \mathscr{R}^M(S_\mu)$ and $p_- < q \le r < p_+$. Note that, for $tD\phi(tB_0D)$ where $\phi \in \mathscr{R}^M(S_\mu)$, we can use Lemma 5.2.8, and for $\psi(tT)$ where $\psi \in \Psi_\sigma^\tau(S_\mu)$ such that $\tau > n\left(\frac{1}{p_-} - \frac{1}{p_+}\right)$ and σ is large enough, we can use Lemma 5.2.9, to obtain such large order $L^q - L^r$ off-diagonal decay for $p_- < q \le r < p_+$. Hence the above arguments also apply to prove the remaining claims. We omit the details. $\square$

Remark 5.5.5. The temporal weight t^{-1} involved in V^p in both Lemma 5.5.3 and Lemma 5.5.4 is not important. Indeed, for $V_\beta^p := L_x^p\left(L^2\left(\mathbb{R}_+, t^\beta dt\right)\right)$, $\beta \in \mathbb{R}$, proving the analogue of Lemma 5.5.3 for V_β^p by using Kalton-Weis multiplier theorem only needs to use the fact that $L^2\left(\mathbb{R}_+, t^\beta dt\right)$ is a Hilbert space. This can be specifically verified by repeating the proof of Lemma 10.3 of [FMP14] with V^p replaced by V_β^p. Also for Lemma 5.5.4, the weighted result follows from the current (-1)-weighted statement.

5.5.3 Intermediate weights

Here we prove, in the functional calculus interval $p \in (p_-, p_+)$, Theorem 5.1.1 in the case of intermediate weights, namely, when $\beta \in (-1, 1)$.

We assume $F \in T_\beta^p$, and also in $L_c^2(\mathbb{R}_+^{1+n}; \mathbb{C}^N)$.

Upon using the singular-regular decompositions (5.5.1) for $\mathbf{S}_{DB_0}^+$ and Lemma 5.5.1, it remains to show for $j = 1, \cdots, \ell$ that

$$\left\| \mathbf{V}_{DB_0}^{+,+,j}(\mathbf{E}F) \right\|_{T_\beta^p} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T_\beta^p}.$$

The proof is the same for each j and we do it for $j = 1$. Since $\beta < 1$, we write [5]

$$\begin{aligned}\mathbf{V}_{DB_0}^{+,+}(\mathbf{E}F)_t &= \int_0^t \left(\frac{s}{t}\right) tDe^{-t|B_0D|}e^{-s|B_0D|}\chi^+(B_0D)(\mathbf{E}F)_s \frac{ds}{s} \\ &= \frac{1}{t^{\frac{1+\beta}{2}}} \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|}e^{-s|B_0D|}\chi^+(B_0D)s^{\frac{1+\beta}{2}}(\mathbf{E}F)_s \frac{ds}{s}.\end{aligned}$$

5. This is a weighted version of Schur type integrals as considered in Lemma 5.5.3. It is well defined when $\mathbf{E}F \in T_\beta^p$ as shown in the following arguments.

Using the tent space isometry

$$F \in T^p_\beta \leftrightarrow \widetilde{F} := s^{\frac{1+\beta}{2}} F_s \in T^p,$$

it further reduces to the boundedness

$$\left\| \widetilde{\mathbf{V}}^{+,+}_{DB_0} \left(\mathbf{E}\widetilde{F}\right) \right\|_{T^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p},$$

where

$$\widetilde{\mathbf{V}}^{+,+}_{DB_0} (\mathbf{E}\widetilde{F})_t = \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|} e^{-s|B_0D|} \chi^+(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}.$$

Similarly, upon using the singular-regular decompositions (5.5.2) for $\mathbf{S}^-_{DB_0}$ and Lemma 5.5.2, and using the similarity in $\mathbf{V}^{-,-,j}_{DB_0}$, it suffices to show

$$\left\| \mathbf{V}^{-,-}_{DB_0} (\mathbf{E}F) \right\|_{T^p_\beta} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_\beta}.$$

Since $\beta > -1$, we write

$$\begin{aligned} \mathbf{V}^{-,-}_{DB_0} (\mathbf{E}F)_t &= \int_t^\infty \left(\frac{t}{s}\right) sDe^{-s|B_0D|} e^{-t|B_0D|} \chi^-(B_0D) (\mathbf{E}F)_s \frac{ds}{s} \\ &= \frac{1}{t^{\frac{1+\beta}{2}}} \int_t^\infty \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} sDe^{-s|B_0D|} e^{-t|B_0D|} \chi^-(B_0D) s^{\frac{1+\beta}{2}} (\mathbf{E}F)_s \frac{ds}{s}. \end{aligned}$$

Using the same isometry in tent space it further reduces to the boundedness

$$\left\| \widetilde{\mathbf{V}}^{-,-}_{DB_0} \left(\mathbf{E}\widetilde{F}\right) \right\|_{T^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p},$$

where

$$\widetilde{\mathbf{V}}^{-,-}_{DB_0} (\mathbf{E}F)_t = \int_t^\infty \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} sDe^{-s|B_0D|} e^{-t|B_0D|} \chi^-(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}.$$

We assume $\mathbf{E}\widetilde{F} \in T^p$, and also in $L^2_c\left(\mathbb{R}^{1+n}_+; \mathbb{C}^N\right)$ in the following arguments.

Subcase $p_- < p < 2$. It suffices to look at $\widetilde{\mathbf{V}}^{+,+}_{DB_0}$. To see this, using the abstract operator $\widetilde{\mathbf{E}} = B_0^{-1}\mathbb{P}_{B_0D}\mathbf{E}$ defined before, we can write

$$\begin{aligned} \widetilde{\mathbf{V}}^{-,-}_{DB_0} \left(\mathbf{E}\widetilde{F}\right)_t = &\int_0^\infty \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|} sDe^{-s|B_0D|} \chi^-(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s} \\ &- \int_0^t \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|} sDe^{-s|B_0D|} \chi^-(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s} \end{aligned}$$

$$
\begin{aligned}
&= -(t|DB_0|)^{\frac{1+\beta}{2}} e^{-t|DB_0|} \int_0^\infty (s|DB_0|)^{\frac{1-\beta}{2}} e^{-s|DB_0|} \chi^-(DB_0) \left(\widetilde{\mathbf{E}}\widetilde{F}\right)_s \frac{ds}{s} \\
&\qquad - \int_0^t \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|} sDe^{-s|B_0D|} \chi^-(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s} \\
&= -(t|DB_0|)^{\frac{1+\beta}{2}} e^{-t|DB_0|} \int_0^\infty (s|DB_0|)^{\frac{1-\beta}{2}} e^{-s|DB_0|} \chi^-(DB_0) \left(\widetilde{\mathbf{E}}\widetilde{F}\right)_s \frac{ds}{s} \\
&\qquad - \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|} e^{-s|B_0D|} \chi^-(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s} \\
&=: I + II.
\end{aligned}
$$

Lemma 5.4.1 takes care of I and bounds it in T^p, since $\widetilde{F}/s^{\frac{1+\beta}{2}} \in T^p_\beta$. For II, we use arguments similar to what we do for $\widetilde{\mathbf{V}}^{+,+}_{DB_0}$ next.

Now for $\widetilde{\mathbf{V}}^{+,+}_{DB_0}$ and for L large enough, using Lemma 5.2.10 there exists $\phi^+ \in H^\infty(S_\mu)$ such that $\phi^+(tB_0D)$ coincides with $e^{-t|B_0D|}$ on $\chi^+(B_0D)\overline{\mathbf{R}_2(B_0D)}$ and $\{tD\phi^\pm(tB_0D)\}_{t>0}$ has $L^q - L^r$ off-diagonal decay of order L, with $p_- < q \le r < p_+$. Thus, we write

$$
\widetilde{\mathbf{V}}^{+,+}_{DB_0}\left(\mathbf{E}\widetilde{F}\right)_t = tD\phi^+(tB_0D) \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} e^{-s|B_0D|} \chi^+(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}.
$$

With this representation, we first use the embedding (see [AHM12])

$$
\mathbf{E}\widetilde{F} \in T^p \hookrightarrow V^p, \quad p < 2.
$$

Then we use the boundedness of $\chi^+(B_0D)$ from $L^p(\mathbb{R}^n; H) = V^p$ to $\overline{\mathbf{R}_p(B_0D)}(\mathbb{R}^n; H)$, where we recall that $H = L^2(\mathbb{R}_+, t^{-1}dt)$, and use the R-boundedness of $\{e^{-s|B_0D|}\}_{s>0}$ in $\overline{\mathbf{R}_p(B_0D)}$ (due to Lemma 5.2.3 and $p \in (p_-, p_+)$), so that we can apply Lemma 5.5.3 to have the estimates

$$
\left\| \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} e^{-s|B_0D|} \chi^+(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s} \right\|_{V^p} \lesssim \left\|\mathbf{E}\widetilde{F}\right\|_{V^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p}.
$$

Finally, applying Lemma 5.5.4 to the operator family $tD\phi^+(tB_0D)$ we have

$$
\left\|\widetilde{\mathbf{V}}^{+,+}_{DB_0}\left(\mathbf{E}\widetilde{F}\right)\right\|_{T^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p}.
$$

This way we get back to the (weighted) tent space estimates for $\mathbf{V}^{+,+}_{DB_0}(\mathbf{E}F)$.

Subcase $2 < p < p_+$**.** It suffices to look at $\widetilde{\mathbf{V}}^{-,-}_{DB_0}$. To see this, again using the abstract operator $\widetilde{\mathbf{E}} = B_0^{-1}\mathbb{P}_{B_0D}\mathbf{E}$, we have

$$
\widetilde{\mathbf{V}}^{+,+}_{DB_0}(\mathbf{E}\widetilde{F})_t = \int_0^\infty \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|} e^{-s|B_0D|} \chi^+(B_0D) \left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}
$$

$$
\begin{aligned}
&-\int_t^\infty \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|}e^{-s|B_0D|}\chi^+(B_0D)\left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}\\
&= (t|DB_0|)^{\frac{1+\beta}{2}} e^{-t|DB_0|}\int_0^\infty (s|DB_0|)^{\frac{1-\beta}{2}} e^{-s|DB_0|}\chi^+(DB_0)\left(\widetilde{\mathbf{E}}\widetilde{F}\right)_s \frac{ds}{s}\\
&\quad -\int_t^\infty \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} tDe^{-t|B_0D|}e^{-s|B_0D|}\chi^+(B_0D)\left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s},\\
&= (t|DB_0|)^{\frac{1+\beta}{2}} e^{-t|DB_0|}\int_0^\infty (s|DB_0|)^{\frac{1-\beta}{2}} e^{-s|DB_0|}\chi^+(DB_0)\left(\widetilde{\mathbf{E}}\widetilde{F}\right)_s \frac{ds}{s}\\
&\quad -\int_t^\infty \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|}sDe^{-s|B_0D|}\chi^+(B_0D)\left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}\\
&=: III + IV.
\end{aligned}
$$

Lemma 5.4.1 takes care of III and bounds it in T^p, since $\widetilde{F}/s^{\frac{1+\beta}{2}} \in T^p_\beta$. For IV, we use arguments similar to what we do for $\widetilde{\mathbf{V}}^{-,-}_{DB_0}$ next.

Now for $\widetilde{\mathbf{V}}^{-,-}_{DB_0}$ and for L large enough, using Lemma 5.2.10 there exists $\phi^- \in H^\infty(S_\mu)$ such that $\phi^-(sB_0D)$ coincides with $e^{-s|B_0D|}$ on $\chi^-(B_0D)\overline{\mathbf{R}_2(B_0D)}$ and $\{sD\phi^-(sB_0D)\}_{s>0}$ has $L^q - L^r$ off-diagonal decay of order L, with $p_- < q \le r < p_+$. Thus, we write

$$
\widetilde{\mathbf{V}}^{-,-}_{DB_0}\left(\mathbf{E}\widetilde{F}\right)_t = \int_t^\infty \left(\frac{t}{s}\right)^{\frac{1+\beta}{2}} e^{-t|DB_0|}\chi^-(DB_0)sD\phi^-(sB_0D)\left(\mathbf{E}\widetilde{F}\right)_s \frac{ds}{s}.
$$

Here we used the functional calculus relation

$$
sD\phi^-(sB_0D)\chi^-(B_0D) = \chi^-(DB_0)sD\phi^-(sB_0D).
$$

With this representation, we first apply Lemma 5.5.4 to the operator family $\{sD\phi^-(sB_0D)\}_{s>0}$ to have the estimate

$$
\left\|\left(sD\phi^-(sB_0D)\left(\mathbf{E}\widetilde{F}\right)_s\right)(\cdot)\right\|_{V^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p}.
$$

Then, with $H = L^2\left(\mathbb{R}_+, t^{-1}dt\right)$, we can use the boundedness of $\chi^+(DB_0)$ from $L^p(\mathbb{R}^n;H)$ to $\overline{\mathbf{R}_p(DB_0)}(\mathbb{R}^n;H)$, and use the R-boundedness of $\{e^{-s|DB_0|}\}_{s>0}$ in $\overline{\mathbf{R}_p(DB_0)}$ (due to Lemma 5.2.3 and $p \in (p_-, p_+)$), so that we can apply Lemma 5.5.3 to have

$$
\left\|\widetilde{\mathbf{V}}^{-,-}_{DB_0}\left(\mathbf{E}\widetilde{F}\right)\right\|_{V^p} \lesssim \left\|\left(sD\phi^-(sB_0D)\left(\mathbf{E}\widetilde{F}\right)_s\right)(\cdot)\right\|_{V^p} \lesssim \|\mathbf{E}\|_\infty \left\|\widetilde{F}\right\|_{T^p}.
$$

Finally, we use the embedding (see [AHM12])

$$
\widetilde{\mathbf{V}}^{-,-}_{DB_0}\left(\mathbf{E}\widetilde{F}\right) \in V^p \hookrightarrow T^p, \quad p > 2.
$$

This way we get back to the (weighted) tent space estimates for $\mathbf{V}^{-,-}_{DB_0}(\mathbf{E}F)$.

5.5.4 Endpoint weights

Here we prove, again in the functional calculus interval, Theorem 5.1.1 in the case of endpoint weight, namely, when $\beta = -1$. In other words, we prove the claims of Theorem 5.1.1 in part 2) for $p \in (p_-, p_+)$ and $\widetilde{p} \in (\widetilde{p}_-, \widetilde{p}_+)$. Recall that we assume $\mathbf{E} \in \widetilde{T}^\infty$, and the integral operators $\mathbf{S}^\pm_{DB_0}$ act on the pointwise product $\mathbf{E}F$.

Again, with the singular-regular decompositions (5.5.1)-(5.5.2) and Lemmas 5.5.1-Lemma 5.5.2 at hand, it suffices to deal with the regular parts $\mathbf{V}^{+,+}_{DB_0}$ and $\mathbf{V}^{-,-}_{DB_0}$.

We prove the second claim in part 2) first.

Part 2)-(ii). Here $sF_s(\cdot) \in T^p$, and also in $L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N)$, with

$$p = (\widetilde{p})' \in \left((\widetilde{p}_+)', (\widetilde{p}_-)'\right) = (p_-, p_+)$$

by formal calculations.

Subcase $2 < \widetilde{p} < \widetilde{p}_+$**.** Hence $p_- < p < 2$. Using the assumption $\mathbf{E} \in \widetilde{T}^\infty$ and the claim in part II) of Lemma 5.4.3 we have

$$t\int_0^\infty De^{-(t+s)|B_0D|}\chi^-(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad T^p.$$

Hence, estimating the regular part $\mathbf{V}^{-,-}_{DB_0}(\mathbf{E}F) \in \left(T^{\widetilde{p}}\right)'$ reduces to estimating

$$t\int_0^t De^{-(t+s)|B_0D|}\chi^-(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad T^p.$$

Note that this is an estimate similar to $t\mathbf{V}^{+,+}_{DB_0}(\mathbf{E}F) \in T^p$.

Now for L large enough, using Lemma 5.2.10 there exist $\phi^\pm \in H^\infty(S_\mu)$ such that $\phi^\pm(tB_0D)$ coincides with $e^{-t|B_0D|}$ on $\chi^\pm(B_0D)\overline{\mathbf{R}_2(B_0D)}$ and $\{tD\phi^\pm(tB_0D)\}_{t>0}$ has L^q–L^r off-diagonal decay of order L, with $p_- < q \le r < p_+$. This way we can write

$$\begin{aligned}
\mathbf{V}^{+,\pm}_{DB_0}(\mathbf{E}F)_t &= \int_0^t De^{-(t+s)|B_0D|}\chi^\pm(B_0D)(\mathbf{E}F)_s\,ds \\
&= \frac{1}{t}\int_0^t (tD)e^{-(t+s)|B_0D|}\chi^\pm(B_0D)(s\mathbf{E}F)_s\,\frac{ds}{s} \\
&= \frac{1}{t}\,tD\phi^\pm(tB_0D)\int_0^t e^{-s|B_0D|}\chi^\pm(B_0D)(s\mathbf{E}F)_s\,\frac{ds}{s}.
\end{aligned}$$

A direct use of Lemma 5.5.3 causes a problem. But note that we have Remark 5.5.5 to take care of the weighted issues. Now we can also write

$$\mathbf{V}^{+,\pm}_{DB_0}(\mathbf{E}F)_t = tD\phi^{\pm}(tB_0D)\int_0^t \left(\frac{s}{t}\right) e^{-s|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\frac{ds}{s}.$$

For simplicity, let

$$T^p_{+1} = \{F : sF_s \in T^p\} \text{ and } V^p_{+1} = \{F : sF_s \in V^p\},$$

both associated with the natural norms. Also let

$$H_{+1} = \{h : sh_s \in H\}.$$

Recall that $H = L^2(\mathbb{R}_+, dt/t)$. Applying first the weighted embedding (the unweighted one in [AHM12] extends to this setting)

$$\mathbf{E}F \in T^p_{+1} \hookrightarrow V^p_{+1}, \quad p \le 2,$$

and then using the $V^p_{+1} = L^p(\mathbb{R}^n; H_{+1}) \to \overline{\mathbf{R}_p(B_0D)}(\mathbb{R}^n; H_{+1})$ boundedness of $\chi^{\pm}(B_0D)$ and the R-boundedness of $\{e^{-s|DB_0|}\}_{s>0}$ in $\overline{\mathbf{R}_p(DB_0)}$, we have

$$\left\| \int_0^t \left(\frac{s}{t}\right) e^{-s|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\frac{ds}{s}\right\|_{V^p_{+1}} \lesssim \|\mathbf{E}F\|_{V^p_{+1}} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_{+1}}.$$

Finally, applying (weighted) Lemma 5.5.4 to the operator family $tD\phi^+(tB_0D)$ we have

$$\begin{aligned}
&\left\| t\int_0^t De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\,ds\right\|_{T^p} \\
&\qquad = \left\| \int_0^t De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\,ds\right\|_{T^p_{+1}} \\
&\qquad \lesssim \left\| \int_0^t \left(\frac{s}{t}\right) e^{-s|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\frac{ds}{s}\right\|_{V^p_{+1}} \lesssim \|\mathbf{E}\|_\infty \|F\|_{T^p_{+1}}.
\end{aligned}$$

Note that $\|F\|_{T^p_{+1}} = \|F\|_{(T^{\tilde{p}})'}$. This way we get back to the (weighted) tent space estimates for $\mathbf{V}^{+,+}_{DB_0}(\mathbf{E}F)$ and $\mathbf{V}^{-,-}_{DB_0}(\mathbf{E}F)$, hence the tent space estimate for $\mathbf{S}^{\pm}_{DB_0}(\mathbf{E}F)$.

Subcase $\tilde{p}_- < \tilde{p} < 2$. Hence $2 < p < p_+$. Upon using part II) of Lemma 5.4.3, together with the assumption $\mathbf{E} \in \widetilde{T}^\infty$, estimating the regular part $\mathbf{V}^{+,+}_{DB_0}$ reduces to estimating

$$t\int_t^\infty De^{-(t+s)|B_0D|}\chi^{+}(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad T^p.$$

Now for L large enough, using Lemma 5.2.10 there exist $\phi^{\pm} \in H^{\infty}(S_\mu)$ such that $\phi^{\pm}(sB_0D)$ coincides with $e^{-s|B_0D|}$ on $\chi^{\pm}(B_0D)\overline{\mathbf{R}_2(B_0D)}$ and $\{sD\phi^{\pm}(sB_0D)\}_{s>0}$ has $L^q - L^r$ off-diagonal decay of order L, with $p_- < q \le r < p_+$. This way we can write

$$\begin{aligned}
\mathbf{V}^{-,\pm}_{DB_0}(\mathbf{E}F)_t &= \int_t^\infty De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(s\mathbf{E}F)_s \frac{ds}{s} \\
&= \frac{1}{t}\int_t^\infty \left(\frac{t}{s}\right) sDe^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(s\mathbf{E}F)_s \frac{ds}{s} \\
&= \frac{1}{t}\int_t^\infty \left(\frac{t}{s}\right) e^{-t|DB_0|}\chi^{\pm}(DB_0)(sD)\phi^{\pm}(sB_0D)(s\mathbf{E}F)_s \frac{ds}{s}.
\end{aligned}$$

Applying Lemma 5.5.4, we have

$$\left\| \left((sD)\phi^{-}(sB_0D)\,(s\mathbf{E}F)_s\right)(\cdot)\right\|_{V^p} \lesssim \|\mathbf{E}\|_\infty \|sF_s\|_{T^p}.$$

Applying Lemma 5.5.3, we have

$$\left\| \int_t^\infty \left(\frac{t}{s}\right) e^{-t|DB_0|}\chi^{\pm}(DB_0)(sD)\phi^{\pm}(sB_0D)(s\mathbf{E}F)_s \frac{ds}{s}\right\|_{V^p} \lesssim \|\mathbf{E}\|_\infty \|sF_s\|_{T^p}.$$

Using the embedding $V^p \hookrightarrow T^p$ for $p \ge 2$ we then finish the proof.

Part 2)-(i). Here $F \in \widetilde{T}^p$. As $\mathbf{E} \in \widetilde{T}^\infty$, this implies $\mathbf{E}F \in T^p$.

In justifying the actions of $\chi^{\pm}(B_0D)$ on $(\mathbf{E}F)$ as below, we replace $\mathbf{E}F$ by a generic $G \in T^p$ which is also assumed to be in $L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N)$.

Subcase $p_- < p < 2$**.** Using part I) of Lemma 5.4.3 we know that

$$\int_0^\infty De^{-(t+s)|B_0D|}\chi^{-}(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad \widetilde{T}^p.$$

Then, estimating the regular part $\mathbf{V}^{-,-}_{DB_0}(\mathbf{E}F)$ in $\widetilde{T}^p$ reduces to estimating

$$\int_0^t De^{-(t+s)|B_0D|}\chi^{-}(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad \widetilde{T}^p \quad \text{or} \quad \text{in} \quad T^p,$$

since we have the natural embedding $T^p \hookrightarrow \widetilde{T}^p$ (see [AA11b]). Now we have

$$\mathbf{V}^{+,\pm}_{DB_0}(\mathbf{E}F)_t = \int_0^t De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\,ds$$

$$= \int_0^t \left(\frac{s}{t}\right)(tD)e^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s \frac{ds}{s}.$$

The remaining arguments are similar to the part 2)-(ii).

Subcase $2 < p < p_+$. Upon using part I) of Lemma 5.4.3, estimating the regular part $\mathbf{V}^{+,+}_{DB_0}$ in $\widetilde{T}^p$ reduces to estimating

$$\int_t^\infty De^{-(t+s)|B_0D|}\chi^+(B_0D)(\mathbf{E}F)_s\,ds \quad \text{in} \quad \widetilde{T}^p \quad \text{or} \quad \text{in} \quad T^p.$$

Now it suffices to bound in T^p

$$\mathbf{V}^{-,\pm}_{DB_0}(\mathbf{E}F)_t = \int_t^\infty De^{-(t+s)|B_0D|}\chi^{\pm}(B_0D)(\mathbf{E}F)_s\,ds.$$

Note that for

$$\mathbf{V}^{+,\pm}_{DB_0^*}(G)_t = \int_0^t De^{-(t+s)|B_0^*D|}\chi^{\pm}(B_0^*D)(G)_s\,ds,$$

we can show as in the proof of part 2)-(ii) that when $\widetilde{p}_- < p' < 2$

$$\left\| t\mathbf{V}^{+,\pm}_{DB_0^*}(G)_t \right\|_{T^{p'}} \lesssim \|sG_s\|_{T^{p'}}.$$

This gives by duality the T^p boundedness of $\mathbf{V}^{-,\pm}_{DB_0}$ for $2 < p < p_+$.

5.5.5 Dual claims

The above arguments in the previous subsections conclude the proof of Theorem 5.1.1 for part 1)-(i), and also for part 2) in the functional calculus intervals $p \in (p_-, p_+)$ for DB_0 and $\widetilde{p} \in (\widetilde{p}_-, \widetilde{p}_+)$ for DB_0^*.

Symmetrically, we can conclude by duality the proof for part 1)-(ii) of Theorem 5.1.1 when $\widetilde{p} \in (\widetilde{p}_-, \widetilde{p}_+)$ for DB_0^*. This duality is seen by considering

$$\langle \mathbf{S}_{DB_0}(\mathbf{E}F), G\rangle = \langle (\mathbf{E}F), (\mathbf{S}_{DB_0})^* G\rangle, \tag{5.5.6}$$

with the tent space pairing $\langle\cdot,\cdot\rangle$ again given by (5.1.1). Formally, we have

$$(\mathbf{S}_{DB_0})^*(G)_t = \int_0^t De^{-(t-s)|B_0^*D|}\chi^-(B_0^*D)G_s ds$$
$$- \int_t^\infty De^{-(s-t)|B_0^*D|}\chi^+(B_0^*D)G_s ds,$$

when $G \in T^{\widetilde{p}}_\beta$ also in $L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. We see that $\left(\mathbf{S}_{DB_0}\right)^*(G)$ differs with $\mathbf{S}_{DB_0^*}$ only by the positions of spectral projections.

Note that the restriction of the extrapolation range in p (respectively in $\widetilde{p}$) comes from using R-boundedness and functional calculus in estimating the regular parts $\mathbf{V}^{\pm,\pm}_{DB_0}$ and $\widetilde{\mathbf{V}}^{\pm,\pm}_{DB_0}$ (respectively, in estimating $\mathbf{V}^{\pm,\pm}_{DB_0^*}$ and $\widetilde{\mathbf{V}}^{\pm,\pm}_{DB_0^*}$).

5.6 Extensions of Theorem 5.1.1: extrapolation

To obtain extensions of Theorem 5.1.1 outside the functional calculus intervals, we need consider the following assumptions on the spectral projections $\chi^\pm(DB_0)$:

$$\text{(KT1)}\qquad \begin{gathered} \forall\ \left(p_-\right)_* < p \le p_-,\ \forall\ G \in L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right) \\ \left\|\chi^\pm(DB_0)(DG)\right\|_{T^p} \lesssim \|DG\|_{T^p}. \end{gathered}$$

where D is the Dirac operator. Similarly let

$$\text{(KT2)}\qquad \begin{gathered} \forall\ \left(\widetilde{p}_-\right)_* < p \le \widetilde{p}_-,\ \forall\ G \in L^2_{\text{loc}}\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right) \\ \left\|\chi^\pm\left(DB_0^*\right)(DG)\right\|_{T^p} \lesssim \|DG\|_{T^p}. \end{gathered}$$

Denote by (KT) both (KT1) and (KT2). Here, (KT) designates Kato in Tent spaces. See Section 2.1 of the 2008 El Escorial survey article [AAM10a] (and also [AKM06]) for the illustrations about how the twisted spectral projection

$$\operatorname{sgn}(B_0D) = \chi^+(B_0D) - \chi^+(B_0D)$$

when $B_0 = \begin{bmatrix} I & 0 \\ 0 & A_0 \end{bmatrix}$ and $A_0 \in L^\infty(\mathbb{R}^n;\mathscr{L}(\mathbb{C}^{nm}))$ is related to the Kato square root problem solved by Auscher et al. in [AHL+02].

Note that when T^p in (KT) is replaced by the boundary space $\mathbb{H}^p_D$, and when $G \in L^2_{\text{loc}}(\mathbb{R}^n;\mathbb{C}^{nm})$ such that $DG \in \mathbb{H}^p_D$, then by Lemma 5.3.1 the boundary space version of (KT) always holds. Moreover, (KT) is always true for $p = 2$, and when $DG \in T^2$, one has the functional calculus relations

$$\chi^\pm(DB_0)(DG) = D\chi^\pm(B_0D)(G) \text{ and } \chi^\pm(DB_0^*)(DG) = D\chi^\pm(B_0^*D)(G).$$

We point out that the weighted T^p_β versions of (KT) are also valid once (KT) holds.

For $\beta \in [-1,1)$, recall $p_{*,\beta} = \frac{np}{n+\frac{(1-\beta)}{2}p}$ and write p_* for $p_{*,\beta}$ if $\beta = -1$.

Theorem 5.6.1. *Assume (KT). We have the following results.*

1) Intermediate-weight maximal regularity estimates. Assume $\mathbf{E} \in L^\infty$ *and* $B_0^{-1} \in L^\infty$. *In this case let* $\widetilde{\mathbf{E}} = B_0^{-1}\mathbf{E}$*, which is also in* L^∞.

(i) For $\beta \in (-1,1)$ *and* $\max\left\{(p_-)_{*,\beta}, 1\right\} < p \le p_-$ *we have*

$$\left\| \mathbf{S}_{DB_0}(\mathbf{E}F) \right\|_{T^p_\beta} \lesssim \|\widetilde{\mathbf{E}}\|_\infty \|F\|_{T^p_\beta}. \tag{5.6.1}$$

(ii) For $\beta \in (-1,1)$ *and* $(\widetilde{p}_-)_{*,-\beta} < \widetilde{p} \le \widetilde{p}_-$ *we have*

$$\left\| \mathbf{S}_{DB_0}(\mathbf{E}F) \right\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'} \lesssim \|\widetilde{\mathbf{E}}\|_\infty \|F\|_{\left(T^{\widetilde{p}}_{-\beta}\right)'}. \tag{5.6.2}$$

2) Endpoint-weight maximal regularity estimates. Assume $\mathbf{E} \in \widetilde{T}^\infty$.

(i) For $(p_-)_* < p \le p_-$ *we have*

$$\left\| \mathbf{S}_{DB_0}(\mathbf{E}F) \right\|_{\widetilde{T}^p} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty} \|F\|_{\widetilde{T}^p}. \tag{5.6.3}$$

(ii) For $(\widetilde{p}_-)_* < \widetilde{p} \le \widetilde{p}_-$ *we have*

$$\left\| \mathbf{S}_{DB_0}(\mathbf{E}F) \right\|_{(T^{\widetilde{p}})'} \lesssim \|\mathbf{E}\|_{\widetilde{T}^\infty} \|F\|_{(T^{\widetilde{p}})'}. \tag{5.6.4}$$

In the above theorem, 2)-(i) corresponds to 1)-(i) for $\beta = -1$ and 2)-(ii) corresponds to 1)-(ii) for $\beta = 1$. In the sequel we do not argue separately with respect to the intermediate or endpoint weight claims. Also note that we do not have to precise the multiplication $\mathbf{E}F$, and we can consider the action of $\mathbf{S}_{DB_0}$ on general $G \in T^p_\beta$.

Upon using the singular-regular decompositions on $\mathbf{S}_{DB_0}$ and trace space duality[6] as in last section, and reducing the weight in T^p_β, we are required to treat

$$\widetilde{\mathbf{V}}^{+,\pm}_{DB_0}(G)_t = \chi^\pm(DB_0)\widehat{\mathbf{H}}^+_{DB_0}(G)_t$$

where, for $-1 \le \beta < 1$,

$$\widehat{\mathbf{H}}^+_{DB_0}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} (tD)e^{-t|B_0D|}e^{-s|B_0D|}G_s \frac{ds}{s}$$

when $G \in T^p$ and is also in $L^2_c(\mathbb{R}^{1+n}_+; \mathbb{C}^N)$. Symmetrically, upon using the singular-regular decompositions on $\mathbf{S}_{DB_0}$ and trace space duality, we are required to treat dual operators of $\widetilde{\mathbf{V}}^{-,\pm}_{DB_0}$, and by reducing the weight in $T^{\widetilde{p}}_{-\beta}$, these operators are

$$\widetilde{\mathbf{V}}^{+,\pm}_{DB_0^*}(G)_t = \chi^\pm(DB_0^*)\widehat{\mathbf{H}}^+_{DB_0^*}(G)_t$$

where, for $-1 < \beta \le 1$, hence $-1 \le -\beta < 1$,

$$\widehat{\mathbf{H}}^+_{DB_0^*}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{\frac{1-(-\beta)}{2}} (tD)e^{-t|B_0^*D|}e^{-s|B_0^*D|}G_s \frac{ds}{s}$$

6. Note that using Lemma 5.4.2 requires the assumption $B_0^{-1} \in L^\infty$ and leads to the restriction $p > 1$ in the claim part 1-(i) of Theorem 5.6.1.

$$= \int_0^t \left(\frac{s}{t}\right)^{\frac{1+\beta}{2}} (tD)e^{-t|B_0^*D|}e^{-s|B_0^*D|}G_s\frac{ds}{s}$$

when $G \in T^{\widetilde{p}}$ and is also in $L^2_c(\mathbb{R}^{1+n}_+;\mathbb{C}^N)$. Note that we have already reduced the weighted T^p_β and $T^{\widetilde{p}}_{-\beta}$ estimates to T^p and $T^{\widetilde{p}}$ estimates. As explained in last section, we are reduced to prove the T^p boundedness of $\widetilde{\mathbf{V}}^{+,\pm}_{DB_0}$ for $(p_-)_{*,\beta} < p \le p_-$ and $-1 \le \beta < 1$, and the $T^{\widetilde{p}}$ boundedness of $\widetilde{\mathbf{V}}^{+,\pm}_{DB_0^*}$ for $(\widetilde{p}_-)_{*,-\beta} < \widetilde{p} \le \widetilde{p}_-$ and $-1 < \beta \le 1$.

Note that one can rewrite

$$\widetilde{\mathbf{V}}^{+,\pm}_{DB_0}(G)_t = \chi^\pm(DB_0)\widetilde{\mathbf{H}}^{+,\pm}_{DB_0}(G)_t$$

with

$$\widetilde{\mathbf{H}}^{+,\pm}_{DB_0}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{\frac{1-\beta}{2}} (tD)\phi^\pm((t+s)B_0D)G_s\frac{ds}{s}$$

with $tD\phi^\pm(tB_0D)$ having large order $L^q - L^2$ off-diagonal decay in t whenever $p_- < q < 2$. Similarly, one can rewrite

$$\widetilde{\mathbf{V}}^{+,\pm}_{DB_0^*}(G)_t = \chi^\pm(DB_0^*)\widetilde{\mathbf{H}}^{+,\pm}_{DB_0^*}(G)_t$$

with

$$\widetilde{\mathbf{H}}^{+,\pm}_{DB_0^*}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{\frac{1+\beta}{2}} (tD)\phi^\pm((t+s)B_0^*D)G_s\frac{ds}{s}$$

with $tD\phi^\pm(tB_0^*D)$ having large order $L^{\widetilde{q}} - L^2$ off-diagonal decay in t whenever $\widetilde{p}_- < \widetilde{q} < 2$. Note that we assumed (KT) to ensure the tent space boundedness of the spectral projections outside the functional calculus intervals. Things are then further reduced to consider the operators $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0}$ and $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0^*}$ only.

We argue in the following two subsections with respect to the tent space boundedness of $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0}$. So by the reduction we have already displayed above and also in last section, the proof for part 1)-(i) and 2)-(i) can be concluded. The results for $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0^*}$ follow similarly (noticing the change in weight $\beta \mapsto -\beta$). Hence the proof for part 1)-(ii) and 2)-(ii) is a symmetric adaptation of the proof for part 1)-(i) and 2)-(i).

5.6.1 Extrapolation by analytic interpolation

Here we treat the cases $(p_-)_{*,\beta} \ge 1$, with $-1 \le \beta < 1$.

Following the strategy of [AKMP12], we consider

$$\widetilde{\mathbf{H}}^{+,\pm}_{DB_0,z}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{z} (tD)\phi^\pm((t+s)B_0D)G_s\frac{ds}{s},$$

where $z \in \mathbb{C}$ and $\operatorname{Re} z > 0$, and when $G \in L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. Hence

$$\widetilde{\mathbf{H}}^{+,\pm}_{DB_0} = \widetilde{\mathbf{H}}^{+,\pm}_{DB_0,\frac{1-\beta}{2}}.$$

The arguments in the proof of Theorem 5.1.1 in the functional calculus interval in last section show that

$$\left\|\widetilde{\mathbf{H}}^{+,\pm}_{DB_0,z}(G)\right\|_{T^p} \lesssim e^{|\operatorname{Im} z|}\|G\|_{T^p}, \quad p \in \left(p_-, p_+\right), \quad \operatorname{Re} z > 0.$$

This includes the case $\operatorname{Re} z = \frac{1-\beta}{2}$ since $\beta < 1$.

Now we rewrite

$$\widetilde{\mathbf{H}}^{+,\pm}_{DB_0,z}(G)_t = \int_0^t \left(\frac{s}{t}\right)^{z-1} D\phi^{\pm}\left((t+s)B_0D\right) G_s\, ds,$$

when $G \in T^p$ and also in $L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$. We need to show for $\operatorname{Re} z$ large enough one has T^1 boundedness of $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0,z}$. Indeed, by the arguments in the proofs of Theorem 3.1 and Lemma 3.5 in [AKMP12] (for $m = 1$ and $\beta = -1$ there), the T^1 boundedness of $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0,z}$ follows if one has

$$(\operatorname{Re} z - 1) + 1 + n\left(\frac{1}{q} - \frac{1}{2}\right) > \frac{n}{2}.$$

This means $\operatorname{Re} z > \frac{n}{q'}$. Now from

$$\frac{1-\beta}{2} = (1-\theta)0 + \theta\frac{n}{q'}$$

we obtain $\theta = \frac{(1-\beta)q'}{2n}$. Note that we are in the case

$$\left(p_-\right)_{*,\beta} \geq 1 \iff \frac{(1-\beta)\left(p_-\right)'}{2n} \leq 1,$$

which means $\theta < 1$ since $q > p_-$. Substituting this θ into the interpolation equality

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{1}$$

we further obtain $p = q_{*,\beta}$. Since we can choose q to be arbitrarily close to p_-, the extrapolation range $p \in \left(\left(p_-\right)_{*,\beta}, p_+\right)$ for the boundedness of $\widetilde{\mathbf{H}}^{+,\pm}_{DB_0}$ in T^p follows.

5.6.2 Extrapolation by atomic decompositions

Here we treat the cases $\left(p_-\right)_{*,\beta} < 1$.

We do not reduce the weight in T^p_β and we estimate directly

$$\mathbf{H}^{+,\pm}_{DB_0}(G)_t = \int_0^t D\phi^{\pm}((t+s)B_0D)G_s ds$$

when $G \in T^p_\beta$, and also in $L^2_c\left(\mathbb{R}^{1+n}_+;\mathbb{C}^N\right)$, $-1 \le \beta < 1$.

For any $\left(p_-\right)_{*,\beta} < p \le 1$, we can find $q > p_-$ but close to p_- such that $\left(p_-\right)_{*,\beta} < q_{*,\beta} < p \le 1$. Note that we have the relations

$$q_{*,\beta} < 1 \Longleftrightarrow q' > \frac{2n}{1-\beta} \Longleftrightarrow \frac{1-\beta}{2} + n\left(\frac{1}{q} - \frac{1}{2}\right) > \frac{n}{2}.$$

Therefore, we can use [AKMP12, Theorem 3.1 (2)] (for $m = 1$ there) to estimate $\mathbf{H}^{+,\pm}_{DB_0}$, and due to the large order $L^q - L^2$ off-diagonal decay, we have the T^p_β boundedness of $\mathbf{H}^{+,\pm}_{DB_0}$ for $p \in \left(q_{*,\beta}, 1\right]$, and by interpolation, for $p \in \left(q_{*,\beta}, p_+\right)$. This implies that for any $\left(p_-\right)_{*,\beta} < p < p_+$ we have the T^p_β boundedness of $\mathbf{H}^{+,\pm}_{DB_0}$.

5.7 Cauchy non-Integral Formulas

As mentioned in the Introduction, here we give the PDE-side motivation for the study of the maximal regularity operator $\mathbf{S}_{DB_0}$, together with the composition operator $\mathbf{S}_{DB_0}(\mathbf{E}\cdot)$ with $\mathbf{E}$ a pointwise L^∞ or $\widetilde{T}^\infty$ multiplication $\mathbf{E}$.

5.7.1 Review of first order formalism

We would like to describe informally the new solvability method of elliptic systems which were developed in [AAM10b, AA11b]. This involves a semigroup approach to (non-)autonomous evolution equations with respect to the bisectorial operators DB_0.

Consider the second order, divergence form and complex valued elliptic system

$$(Lu)^\alpha := \sum_{\beta=1}^{m} \operatorname{div} A^{\alpha\beta} \nabla u^\beta = 0, \quad \alpha = 1, \cdots, m \tag{5.7.1}$$

on the upper half space $\mathbb{R}^{1+n}_+ = \mathbb{R}_+ \times \partial\mathbb{R}^{1+n}_+ = (0,\infty) \times \mathbb{R}^n$, with $m, n \in \mathbb{N}_+$.

As for the notation appeared in (5.7.1), for any $(t, y) \in \mathbb{R}^{1+n}_+$ the coefficient elements $A^{\alpha\beta}(t,x) \in \mathbb{C}^{(1+n)\times(1+n)}$, the weak solutions $u(t,x) = (u^\alpha(t,x))^m_{\alpha=1} \in \mathbb{C}^m$, and the full coefficients $A(t,x) = \left(A^{\alpha\beta}(t,x)\right)^m_{\alpha,\beta=1} \in \mathbb{C}^{N\times N}$ where $N = (1+n)m$. In the two full operations div and ∇, we let the partial actions $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial y_i}$, $1 \le i \le n$.

For an m-tuple of vectors $v = \left(v^\alpha_i\right)^{1\le\alpha\le m}_{0\le i\le n}$, denote by $v_\perp$ and $v_\parallel$ the normal and tangential part of v, namely, for $1 \le \alpha \le m$, $(v_\perp)^\alpha_0 = v^\alpha_0$ and $(v_\perp)^\alpha_i = 0$ when $1 \le i \le n$, whereas $\left(v_\parallel\right)^\alpha_i = v^\alpha_i$ when $1 \le i \le n$ and $\left(v_\parallel\right)^\alpha_0 = 0$.

According to this decomposition of m-tuples, we split the matrix

$$\mathscr{L}\left(\mathbb{C}^N\right) \ni A(t,x) = \begin{bmatrix} A_{\perp\perp}(t,x) & A_{\perp\parallel}(t,x) \\ A_{\parallel\perp}(t,x) & A_{\parallel\parallel}(t,x) \end{bmatrix}.$$

Lemma 5.7.1 ([AA11b]). *The pointwise transformation*

$$A=\begin{bmatrix} A_{\perp\perp} & A_{\perp\|} \\ A_{\|\perp} & A_{\|\|}\end{bmatrix} \to B=\widehat{A}=\begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1}A_{\perp\|} \\ A_{\|\perp}A_{\perp\perp}^{-1} & A_{\|\|}-A_{\|\perp}A_{\perp\perp}^{-1}A_{\perp\|}\end{bmatrix} \tag{5.7.2}$$

is a self-inverse bijective transformation of the set of bounded matrices (namely, $L^\infty\left(\mathbb{R}_+^{1+n};\mathscr{L}\left(\mathbb{C}^N\right)\right)$ fuctions) which are strictly accretive on $\mathscr{H}$.

Moreover, the pointwise map

$$g \mapsto F=\left[(Ag)_\perp, g_\|\right]^t,$$

where $\left[(Ag)_\perp, g_\|\right]^t=\begin{bmatrix}(Ag)_\perp \\ g_\|\end{bmatrix}$, gives a one-to-one correspondence, with inverse $g=\left[(BF)_\perp, F_\|\right]^t$, between solutions $g\in L^2_{\mathrm{loc}}\left(\mathbb{R}_+;L^2\left(\mathbb{R}^n;\mathbb{C}^N\right)\right)$ to the $\operatorname{div}-\operatorname{curl}$ *system*

$$\begin{cases}\operatorname{div}(Ag)=0,\\ \operatorname{curl} g=0\end{cases} \tag{5.7.3}$$

and solutions $F\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathscr{H})$ to the generalized Cauchy-Riemann system

$$\partial_t F + DBF = 0. \tag{5.7.4}$$

Here $B\in L^\infty\left(\mathbb{R}_+^{1+n};\mathscr{L}\left(\mathbb{C}^N\right)\right)$ is strictly accretive on $\mathscr{H}$ means for almost every $t>0$, $B_t=B(t,\cdot)$ is strictly accretive on $\mathscr{H}$.

Lemma 5.7.2 ([AA11b]). *Given $B:\mathbb{R}_+^{1+n}\to\mathscr{L}\left(\mathbb{C}^N\right)$ bounded and strictly accretive on $\mathscr{H}$, with the accretivity constant $\widetilde{\kappa}>0$. Then there is a t-independent measurable coefficients B_0, with the requirement $\|B-B_0\|_{\widetilde{T}^\infty}<\infty$, which is uniquely resulted from B, in the sense that if B_0' is another t-independent measurable choice, also with $\|B-B_0'\|_{\widetilde{T}^\infty}<\infty$, then one has $B_0'=B_0$ almost everywhere. Thus determined, B_0 also satisfies*

$$\widetilde{\kappa}\le\widetilde{\kappa}_0\le\|B_0\|_\infty\le\|B\|_\infty, \tag{5.7.5}$$

where $\widetilde{\kappa}_0$ is the accretivity constant of B_0 and $\|\cdot\|_\infty$ is the norm on $\mathbb{R}^n$ or $\mathbb{R}_+^{1+n}$.

Therefore the second order divergence form elliptic system (5.7.1), written for short

$$Lu=\operatorname{div} A\nabla u=0, \tag{5.7.6}$$

reduces to a non-autonomous evolution equation. Applying the above two lemmas, we can rewrite (5.7.4) as

$$\partial_t F_t + DB_0F_t = D(\mathbf{E}F)_t, \quad F_t\in\mathscr{H}, \tag{5.7.7}$$

where $\mathbf{E}=B_0-B$ and B_0 is the unique t-independent coefficients with $\|B-B_0\|_{\widetilde{T}^\infty}<\infty$. Moreover, B_0 is also bounded and strictly accretive on $\mathscr{H}$.

Note that in the second lemma above, we do not claim the uniqueness of the boundary coefficient B_0 under the requirement $\|B-B_0\|_\infty<\infty$ only.

5.7.2 Construction of weak solutions

Let $\beta \in [-1,1]$ so that $\sigma = \frac{1+\beta}{2} \in [0,1]$. It is formal (see for example, the proof of Theorem 8.2 in [AA11b]) to show that F satisfying

$$F_t - \mathbf{S}_{DB_0}(\mathbf{E}F)_t = |DB_0|^{\sigma} e^{-t|DB_0|} h^+, \quad \sigma \in [0,1), \tag{5.7.8}$$

for some $h^+ \in \mathbb{H}^p_D = \mathbb{H}^p_{DB_0}$. It is also formal (see for example, the proof of Theorem 9.2 in [AA11b]) to show that F satisfying

$$F_t - \mathbf{S}_{DB_0}(\mathbf{E}F)_t = D|B_0 D|^{\sigma-1} e^{-t|B_0 D|} \widetilde{h}^+, \quad \sigma \in (0,1], \tag{5.7.9}$$

for some $\widetilde{h}^+ \in \left(\mathbb{H}^{\widetilde{p}}_D\right)' = \left(\mathbb{H}^{\widetilde{p}}_{DB_0^*}\right)'$, is a weak solution to (5.7.7).

Let $\beta \in [-1,1]$. For $p \in (0,\infty)$, let $\mathscr{E}^p_\beta = T^p_\beta$ for $\beta \in (-1,1]$ and $\mathscr{E}^p_{-1} = \widetilde{T}^p$. Let $\mathscr{C}_\beta = L^\infty$ for $\beta \in (-1,1)$, and $\mathscr{C}_\beta = \widetilde{T}^\infty$ for $\beta = \pm 1$.

Theorem 5.7.3. *We have the following results.*

I) Let $\beta \in [-1,1)$ so that $\sigma = \frac{1+\beta}{2} \in [0,1)$.

If $\|\mathbf{E}\|_{\mathscr{C}_\beta}$ is small, for $p \in \left((p_-)_{*,\beta}, p_+\right)$ (we assume (KT) when $p \le p_-$, and when $\beta \neq -1$, we also assume $B_0^{-1} \in L^\infty$ when $p \le p_-$ and assume $p > \max\left\{(p_-)_{*,\beta}, 1\right\}$ in addition)

$$F = \left(I - \mathbf{S}_{DB_0}\mathbf{E}\right)^{-1} |DB_0|^{\sigma} e^{-t|DB_0|} h^+, \quad h^+ \in \mathbb{H}^p_D, \tag{5.7.10}$$

is an $\mathscr{E}^p_\beta$ solution to (5.7.7), and $g \mapsto F = \left[(Ag)_\perp, g_\parallel\right]^t$ is an $\mathscr{E}^p_\beta$ solution to (5.7.6).

II) Let $\beta \in (-1,1]$ so that $\sigma = \frac{1+\beta}{2} \in (0,1]$.

If $\|\mathbf{E}\|_{\mathscr{C}_\beta}$ is small, for $\widetilde{p} \in \left((\widetilde{p}_-)_{*,-\beta}, \widetilde{p}_+\right)$ (we assume (KT) when $\widetilde{p} \le \widetilde{p}_-$, and when $\beta \neq 1$, we also assume $B_0^{-1} \in L^\infty$ when $\widetilde{p} \le \widetilde{p}_-$)

$$F = \left(I - \mathbf{S}_{DB_0}\mathbf{E}\right)^{-1} D|B_0 D|^{\sigma-1} e^{-t|B_0 D|} \widetilde{h}^+, \quad \widetilde{h}^+ \in \left(\mathbb{H}^{\widetilde{p}}_D\right)', \tag{5.7.11}$$

is an $\left(\mathscr{E}^{\widetilde{p}}_{-\beta}\right)'$ solution to (5.7.7), and $g \mapsto F = \left[(Ag)_\perp, g_\parallel\right]^t$ is an $\left(\mathscr{E}^{\widetilde{p}}_{-\beta}\right)'$ solution to (5.7.6).

Proof. From a careful case-by-case verification of the hypotheses in Theorem 5.1.1 and Theorem 5.6.1, by which we have the $\mathscr{E}^p_\beta$ (or $\left(\mathscr{E}^{\widetilde{p}}_{-\beta}\right)'$) boundedness of $\mathbf{S}_{DB_0}\mathbf{E}$, the theorem is a consequence of the above first order formalism. □

When $p = 2$, (5.7.10)-(5.7.11) are obtained in [AA11b] for $\beta = \pm 1$ and in [Ros14] for $-1 < \beta < 1$. Note that the assumption $B_0^{-1} \in L^\infty$ was used in [Ros14]. Here for $-1 < \beta < 1$ we imposed this assumption only when p (respectively, $\widetilde{p}$) is outside the functional calculus interval (p_-, p_+) (respectively, $(\widetilde{p}_-, \widetilde{p}_+)$).

5.8 Appendix. Singular integrals on tent spaces

Here we provide the detailed proofs of some technical singular integral estimates on tent spaces, namely, Lemmas 5.5.1 and Lemma 5.5.2.

The proofs of Lemma 5.5.1 and Lemma 5.5.2 use the change of aperture results in tent spaces. For $a > 0$, we define

$$\mathscr{A}^a(F)(x) := \left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,at)}(y)}{t^n} |F(t,y)|^2 t^\beta \, dt\, dy \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

and we omit a if $a = 1$. Hence via $\mathscr{A}^a$ one can also define a scale of tent spaces. The change of aperture in tent spaces amounts to say the equivalence of tent space quasi-norms for different apertures a. The sharp dependence in a is obtained in [Aus11] by using atomic decompositions of tent spaces. See also [FS72, Tor86, Uch01, HvNP08] for previous results. The result from [Aus11] reads as:

For any $a \geq 1$ *and* $0 < p < \infty$

$$\left\| \mathscr{A}^a(F) \right\|_{L^p} \leq C(n,p) a^{\frac{n}{\min(p,2)}} \| \mathscr{A}(F) \|_{L^p}.$$

Note that in the right hand side of the above estimate $\|\mathscr{A}(F)\|_{L^p} = \|F\|_{T^p_\beta}$.

5.8.1 Proof of Lemma 5.5.1

Recall that $\beta \leq 1$ and

$$\mathbf{S}^{+,\ell}_{DB_0}(H)_t = \int_0^t De^{-(t-s)|B_0D|} \left(I - e^{-2s|B_0D|}\right)^\ell \chi^+(B_0D)(H)_s \, ds.$$

where $\ell \in \mathbb{N}^*$ and $H \in T^p_\beta$ for $0 < p < \infty$, and also in $L^2_c\left(\mathbb{R}^{1+n}_+; \mathbb{C}^N\right)$.

Case $\mathbf{p = 2}$. Since $\beta \leq 1$, the boundedness of $\mathbf{S}^{+,\ell}_{DB_0}$ on T^2_β for $\ell = 1$ was established in [AA11b] (see also [AA11a]). The proof for general ℓ follows by the uniform L^2 boundedness of $\left(I - e^{-2s|B_0D|}\right)^{\ell-1}$.

We are now to treat the tent space extrapolation of $\mathbf{S}^{+,\ell}_{DB_0}$. We use the L^2 theory only and the change of apertures in tent spaces to prove Lemma 5.5.1.

Case $\mathbf{0 < p < \infty}$ and $\mathbf{p \neq 2}$. We adapt the arguments in [AMP12].

Let $H \in L^2_c\left(\mathbb{R}^{1+n}_+; \mathbb{C}^N\right)$. Given $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and $j \in \mathbb{N}_+$, we consider

$$C_j(x,t) = \begin{cases} B(x,t) & \text{if } j = 0, \\ B\left(x, 2^j t\right) \setminus B\left(x, 2^{j-1} t\right) & \text{if } j \geq 1. \end{cases}$$

We write

$$\left\|\mathbf{S}^{+,\ell}_{DB_0}(H)\right\|_{T^{p,2}} \le \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} I_{k,j} + \sum_{j=0}^{\infty} J_j$$

with

$$I_{k,j} = \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,t)}(y)}{t^n}\left|\int_{2^{-k-1}t}^{2^{-k}t} \frac{\psi^{+,\ell}_{DB_0}(t,s)}{(t-s)}\left(\mathbf{1}_{C_j(x,4t)}H_s\right)(y)ds\right|^2 t^\beta dy\,dt\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}},$$

$$J_j = \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,t)}(y)}{t^n}\left|\int_{t/2}^{t} \frac{\psi^{+,\ell}_{DB_0}(t,s)}{(t-s)}\left(\mathbf{1}_{C_j(x,4s)}H_s\right)(y)ds\right|^2 t^\beta dy\,dt\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}},$$

where

$$\begin{aligned}\psi^{+,\ell}_{DB_0}(t,s) &= (t-s)D\chi^+(B_0D)e^{-(t-s)|B_0D|}\left(I-e^{-2s|B_0D|}\right)^\ell\\ &= \left(\frac{s}{t-s}\right)^\ell (t-s)D\chi^+(B_0D)\left((t-s)|B_0D|\right)^\ell e^{-(t-s)|B_0D|}\frac{\left(I-e^{-2s|B_0D|}\right)^\ell}{(sB_0D)^l}.\end{aligned}$$

Fixing $j\ge 0$, $k\ge 1$ we first estimate $I_{k,j}$ as follows. For fixed $x\in\mathbb{R}^n$,

$$\begin{aligned}&\int_0^\infty\int_{B(x,t)}\left|\int_{2^{-k-1}t}^{2^{-k}t}\frac{\psi^{+,\ell}_{DB_0}(t,s)}{(t-s)}\left(\mathbf{1}_{C_j(x,4t)}H_s\right)(y)\,ds\right|^2 t^\beta\frac{dy\,dt}{t^n}\\
&\lesssim \int_0^\infty\int_{2^{-k-1}t}^{2^{-k}t} 2^{-k}t\left(\int_{B(x,t)}\left|\psi^{+,\ell}_{DB_0}(t,s)\left(\mathbf{1}_{C_j(x,4t)}H_s\right)(y)\right|^2 dy\right)t^\beta\frac{ds\,dt}{t^{n+2}}\\
&\lesssim \int_0^\infty\int_{2^{-k-1}t}^{2^{-k}t} 2^{-k}t\left(\frac{s}{t-s}\right)^{2\ell}\left(1+\frac{2^jt}{t-s}\right)^{-2\ell}\left\|\mathbf{1}_{B(x,2^{j+2}t)}H_s\right\|_2^2 t^\beta\frac{ds\,dt}{t^{n+2}}\\
&\lesssim 2^{-(2l+1)}2^{-2j\ell}\int_0^\infty\left(\int_{2^ks}^{2^{k+1}s}t^\beta\frac{dt}{t^{n+1}}\right)\left\|\mathbf{1}_{B(x,2^{j+k+3}s)}H_s\right\|_2^2\,ds\\
&\lesssim 2^{-k(2l+1+n-\beta)}2^{-2j\ell}\int_0^\infty\left\|\mathbf{1}_{B(x,2^{j+k+3}s)}H_s\right\|_2^2 s^{\beta-n}ds.\end{aligned}$$

In the first inequality, we use Cauchy-Schwarz inequality for the integral with respect to s, the fact that $t-s\sim t$ for $s\in\cup_{k\ge1}\left[2^{-k-1}t,2^{-k}t\right]\subset[0,\frac{t}{2}]$ and Fubini's theorem to exchange the integral in s and the integral in y. The next inequality follows from the off-diagonal estimate verified by $\psi^{+,\ell}_{DB_0}(t,s)$ (and local coercivity inequality) and again the fact that $t-s\sim t$. By the change of angle in tent spaces this gives

$$I_{k,j}\lesssim 2^{-k\left(\frac{1}{2}(2l+n+1-\beta)-\frac{n}{\tau}\right)}2^{-j\left(\ell-\frac{n}{\tau}\right)}\|H\|_{T^p_\beta},$$

where $\tau = \min(p,2)$. It follows that

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} I_{k,j} \lesssim \|H\|_{T^p_\beta}$$

since, for any $p > \frac{n}{n+1}$, $\ell \geq n+1 > \frac{n}{\tau}$, and for $\beta \leq 1$, $2\ell + (1-\beta) + n > \frac{2n}{\tau}$ [7].

We now turn to J_0 and remark that

$$J_0 \leq \left(\int_{\mathbb{R}^n} J_0(x)^{\frac{p}{2}}\, dx \right)^{\frac{1}{p}},$$

where

$$J_0(x) = \iint_{\mathbb{R}^{1+n}_+} \left| \int_{t/2}^{t} \frac{\psi^{+,\ell}_{DB_0}(t,s)}{(t-s)} \left(e^{-s|DB_0|} G_s\right)(y)\, ds \right|^2 t^{\beta-n}\, dy\, dt$$

with

$$G(s,y) = \mathbf{1}_{B(x,4t)}(y) H(s,y).$$

The inside integral can be rewritten as

$$\mathbf{S}^{+,\ell}_{DB_0}(G)(t,\cdot) - e^{-\frac{t}{2}|B_0 D|}\mathbf{S}^{+,\ell}_{DB_0}(G)(t/2,\cdot).$$

As $\mathbf{S}^{+,\ell}_{DB_0}$ is bounded on $L^2\left(\mathbb{R}^{1+n}_+; t^{\beta-n} dt\, dy\right)$ and $\{e^{-t|B_0 D|}\}_{t\geq 0}$ is uniformly bounded on $L^2(\mathbb{R}^n)$, we get

$$J_0(x) \lesssim \int_0^{\infty} \left\| \mathbf{1}_{B(x,4s)} H_s \right\|_2^2 s^{\beta-n}\, ds.$$

We finally turn to J_j, for $j \geq 1$. For fixed $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_0^{\infty}\int_{\mathbb{R}^n} \mathbf{1}_{B(x,t)}(y) \left| \int_{t/2}^{t} \frac{\psi^{+,\ell}_{DB_0}(t,s)}{(t-s)} \left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y)\, ds \right|^2 t^{\beta-n}\, dy\, dt \\
&\leq \iint_{\mathbb{R}^{1+n}_+} \mathbf{1}_{B(x,t)}(y) \left(\int_{t/2}^{t} \left| \psi^{+,\ell}_{DB_0}(t,s) \left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y) \right| \frac{ds}{t-s} \right)^2 t^{\beta-n}\, dy\, dt \\
&\lesssim \int_0^{\infty}\int_{\mathbb{R}^n} \mathbf{1}_{B(x,t)}(y) \int_{t/2}^{t} \left| \psi^{+,\ell}_{DB_0}(t,s) \left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y) \right|^2 \frac{ds}{(t-s)^2}\, t^{\beta-n+1}\, dy\, dt \\
&\lesssim \int_0^{\infty}\int_{t/2}^{t} (t-s)^{-2} \left(1 + \frac{2^j t}{t-s}\right)^{-2\ell} \left\| \mathbf{1}_{B(x,2^{j+2}s)} H_s \right\|_2^2 s^{\beta-n+1}\, ds\, dt \\
&\lesssim 2^{-2j\ell} \int_0^{\infty} \left(\int_s^{2s} s(t-s)^{-2} \left(1 + \frac{2^j t}{t-s}\right)^{-2} dt \right) \left\| \mathbf{1}_{B(x,2^{j+2}s)} H_s \right\|_2^2 s^{\beta-n}\, ds
\end{aligned}$$

7. Note that for $p > \frac{n}{n+1}$ the summability in k only requires $\beta < n+1$. This includes the case $\beta \leq 1$.

$$\lesssim 2^{-2j\ell}\int_0^\infty \left\|\mathbf{1}_{B(x,2^{j+2}s)}H_s\right\|_2^2 s^{\beta-n}\,ds,$$

where we have used Cauchy-Schwarz inequality in the second inequality, the off-diagonal estimates (and local coercivity inequality) and the fact that $s \le t$ in the third, Fubini's theorem and the fact that $s \ge \frac{t}{2}$ in the fourth, and the change of variable $\sigma = \frac{t}{t-s}$ in the last. An application of the change of angles in tent spaces, then gives

$$J_j \lesssim 2^{-j\ell}2^{j\frac{n}{\tau}}\|H\|_{T^p_\beta} = 2^{-j(\ell-\frac{n}{\tau})}\|H\|_{T^p_\beta},$$

and the proof is concluded by summing the estimates.

Case $\alpha \geq 0$. Pick a ball $B(x_0,r) \subset \mathbb{R}^n$. Let

$$I^2 = \int_{B(x_0,r)}\int_0^r \left|\mathbf{S}^{+,\ell}_{DB_0}(H)(t,x)\right|^2 t^\beta\,dx\,dt.$$

We want to show that $I^2 \lesssim r^{n+2\alpha}\|H\|^2_{T^{\infty,\alpha}_\beta}$. We set

$$I_j^2 = \int_{B(x_0,r)}\int_0^r \left|\mathbf{S}^{+,\ell}_{DB_0}(H^j)(t,x)\right|^2 t^\beta\,dx\,dt$$

where $H^j(s,x) = H(s,x)\mathbf{1}_{C_j(x_0,4r)}(x)\mathbf{1}_{(0,r)}(s)$ for $j \ge 0$ and

$$C_j(x,t) = \begin{cases} B(x,t) & \text{if } j=0,\\ B\left(x,2^j t\right)\setminus B\left(x,2^{j-1}t\right) & \text{otherwise.}\end{cases}$$

Thus by Minkowsky inequality, $I \le \sum I_j$. For I_0 we use the L^2 theory which implies that $\mathbf{S}^{+,\ell}_{DB_0}$ is bounded on $L^2\left(\mathbb{R}^{1+n}_+, t^\beta dt\,dx\right)$ for $\beta \le 1$. Thus

$$I_0^2 \lesssim \int_{B(x_0,4r)}\int_0^r |H(t,x)|^2 t^\beta\,dx\,dt \lesssim r^{n+2\alpha}\|H\|^2_{T^{\infty,\alpha}_\beta}.$$

Next, for $j \ne 0$, we proceed as in [AMP12] to obtain

$$\begin{aligned} I_j^2 \lesssim& \sum_{k=1}^\infty \int_0^r \int_{2^{-k-1}t}^{2^{-k}t} 2^{-k}t\left(\frac{s}{t-s}\right)^{2\ell}\left(1+\frac{2^j r}{t-s}\right)^{-2\ell}\left\|H_s^j\right\|^2_{L^2} t^{\beta-2}\,ds\,dt\\ &+\int_0^r\int_{t/2}^t t(t-s)^{-2}\left(\frac{s}{t-s}\right)^{2\ell}\left(1+\frac{2^j r}{t-s}\right)^{-2\ell}\left\|H_s^j\right\|^2_{L^2} t^\beta\,ds\,dt.\end{aligned}$$

Exchanging the order of integration, and using the fact that $t \sim t-s$ in the first part and that $t \sim s$ in the second, we have the following.

$$I_j^2 \lesssim \sum_{k=1}^\infty 2^{-k(2\ell+1)}2^{-2j\ell}r^{-\ell}\int_0^{2^{-k}r}\int_{2^k s}^{2^{k+1}s} t^{\beta+2\ell-1}\left\|H_s^j\right\|^2_{L^2}dt\,ds$$

$$+\int_0^r\int_s^{2s} r(t-s)^{-2}\left(\frac{r}{t-s}\right)^{2\ell}\left(1+\frac{2^j r}{t-s}\right)^{-2\ell}\left\|H_s^j\right\|_{L^2}^2 s^\beta\,dt\,ds$$

$$\lesssim \sum_{k=1}^{\infty} 2^{-k(2\ell+1)}2^{-2j\ell}\int_0^{2^{-k}r}\left(2^k s\right)^\beta\left\|H_s^j\right\|_{L^2}^2 ds$$

$$+\int_0^r\int_1^\infty (1+2^j\sigma)^{-2\ell}\left\|H_s^j\right\|_{L^2}^2 s^\beta\,d\sigma\,ds$$

$$\lesssim 2^{-2j\ell}\int_0^r\left\|H_s^j\right\|_{L^2}^2 s^\beta\,ds,$$

where we used $\beta \le 1 < n+1 \le 2\ell - n - 1$. We thus have

$$I_j^2 \lesssim 2^{-2j\ell}\left(2^j r\right)^{n+2\alpha}\|H\|_{T_\beta^{\infty,\alpha}}^2,$$

and the condition $\ell+1 > n+2\alpha$ allows us to sum these estimates.

5.8.2 Proof of Lemma 5.5.2

For $1<p<\infty$ and $0\le\alpha<1$, by observation (we omit the details) this claim is a dual version of Lemma 5.5.1, with the duality given by

$$\langle F, G\rangle = \iint_{\mathbb{R}_+^{1+n}} F(t,y)\overline{G(t,y)}\,dt\,dy.$$

We shall only need to treat the case $\frac{n}{n+1} < p \le 1$. To do this, we carry out the proof as that of Lemma 5.5.1 in the case $0<p<\infty$.

Let $H \in L_c^2\left(\mathbb{R}_+^{1+n};\mathbb{C}^N\right)$. Given $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$, and $j\in\mathbb{N}_+$, we consider

$$C_j(x,t) = \begin{cases} B(x,t) & \text{if } j=0,\\ B\left(x,2^j t\right)\setminus B\left(x,2^{j-1}t\right) & \text{if } j\ge 1.\end{cases}$$

We write

$$\left\|\mathbf{S}_{DB_0}^{-,\ell}(H)\right\|_{T^{p,2}} \le \sum_{k=1}^\infty\sum_{j=0}^\infty I_{k,j} + \sum_{j=0}^\infty J_j$$

with

$$I_{k,j} = \left(\int_{\mathbb{R}^n}\left(\iint_{\mathbb{R}_+^{1+n}}\frac{\mathbf{1}_{B(x,t)}(y)}{t^n}\left|\int_{2^k t}^{2^{k+1}t}\frac{\psi_{DB_0}^{-,\ell}(t,s)}{(s-t)}\left(\mathbf{1}_{C_j(x,4t)}H_s\right)(y)\,ds\right|^2 t^\beta\,dy\,dt\right)^{\frac{p}{2}}dx\right)^{\frac{1}{p}},$$

$$J_j = \left(\int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,t)}(y)}{t^n} \left| \int_t^{2t} \frac{\psi^{-,\ell}_{DB_0}(t,s)}{(s-t)} \left(\mathbf{1}_{C_j(x,4s)} H_s \right)(y) ds \right|^2 t^\beta dy\, dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

where

$$\begin{aligned} \psi^{-,\ell}_{DB_0}(t,s) &= (s-t) D\chi^-(B_0D) e^{-(s-t)|B_0D|} \left(I - e^{-2t|B_0D|} \right)^\ell \\ &= \left(\frac{t}{s-t} \right)^\ell (s-t)^{\ell+1} D\chi^-(B_0D)|B_0D|^\ell e^{-(s-t)|B_0D|} \frac{\left(I - e^{-2t|B_0D|} \right)^\ell}{(tB_0D)^\ell}. \end{aligned}$$

Fixing $j \geq 0$, $k \geq 1$ we first estimate $I_{k,j}$ as follows. For fixed $x \in \mathbb{R}^n$,

$$\begin{aligned} &\int_0^\infty \int_{B(x,t)} \left| \int_{2^k t}^{2^{k+1}t} \frac{\psi^{-,\ell}_{DB_0}(t,s)}{(s-t)} \left(\mathbf{1}_{C_j(x,4t)} H_s \right)(y)\, ds \right|^2 t^\beta \frac{dy\,dt}{t^n} \\ &\lesssim \int_0^\infty \int_{2^k t}^{2^{k+1}t} 2^{-k} \left(\int_{B(x,t)} \left| \psi^{-,\ell}_{DB_0}(t,s) \left(\mathbf{1}_{C_j(x,4t)} H_s \right)(y) \right|^2 dy \right) t^\beta \frac{ds\,dt}{t^{n+1}} \\ &\lesssim \int_0^\infty \int_{2^k t}^{2^{k+1}t} 2^{-k} \left(\frac{t}{s-t} \right)^{2\ell} \left(1 + \frac{2^j t}{t} \right)^{-2\ell} \left\| \mathbf{1}_{B(x,2^{j+2}t)} H_s \right\|_2^2 t^\beta \frac{ds\,dt}{t^{n+1}} \\ &\lesssim 2^{-(2\ell+1)k} 2^{-2j\ell} \int_0^\infty \left(\int_{2^{-(k+1)}s}^{2^{-k}s} t^\beta \frac{dt}{t^{n+1}} \right) \left\| \mathbf{1}_{B(x,2^{j-k+2}s)} H_s \right\|_2^2 ds \\ &\lesssim 2^{-k(2\ell+1+\beta-n)} 2^{-2j\ell} \int_0^\infty \left\| \mathbf{1}_{B(x,2^{j-k+2}s)} H_s \right\|_2^2 s^{\beta-n} ds. \end{aligned}$$

In the first inequality, we use Cauchy-Schwarz inequality for the integral with respect to s and Fubini's theorem to exchange the integral in s and the integral in y. The next inequality follows from the off-diagonal estimate verified by $\psi^{-,\ell}_{DB_0}(t,s)$ (and local coercivity inequality). By the change of apertures in tent spaces this gives

$$I_{k,j} \lesssim 2^{-k\left(\frac{1}{2}(2\ell+(1+\beta)-n)+\frac{n}{\tau}\right)} 2^{-j\left(\ell-\frac{n}{\tau}\right)} \|H\|_{T^p_\beta},$$

where $\tau = \min(p,2)$. It follows that

$$\sum_{k=1}^\infty \sum_{j=0}^\infty I_{k,j} \lesssim \|H\|_{T^p_\beta}$$

since, for any $p > \frac{n}{n+1}$, $\ell \geq n+1 > \frac{n}{\tau}$, and for $\beta \geq -1$, $2\ell + (1+\beta) - n > -\frac{2n}{\tau}$.

We now turn to J_0 and remark that

$$J_0 \leq \left(\int_{\mathbb{R}^n} J_0(x)^{\frac{p}{2}} dx \right)^{\frac{1}{p}},$$

where

$$J_0(x) = \iint_{\mathbb{R}^{1+n}_+} \left| \int_t^{2t} \frac{\psi^{-,\ell}_{DB_0}(t,s)}{(t-s)} \left(e^{-s|DB_0|} G_s\right)(y)\,ds \right|^2 t^{\beta-n}\,dy\,dt$$

with

$$G(s,y) = \mathbf{1}_{B(x,4t)}(y) H(s,y).$$

The inside integral can be rewritten as

$$\mathbf{S}^{+,\ell}_{DB_0}(G)(t,\cdot) - e^{-\frac{t}{2}|B_0D|}\mathbf{S}^{+,\ell}_{DB_0}(G)(t/2,\cdot).$$

As $\mathbf{S}^{+,\ell}_{DB_0}$ is bounded on $L^2\left(\mathbb{R}^{1+n}_+; t^{\beta-n}dt\,dy\right)$ and $\left\{e^{-t|B_0D|}\right\}_{t\geq 0}$ is uniformly bounded on $L^2(\mathbb{R}^n)$, we get

$$J_0(x) \lesssim \int_0^\infty \left\|\mathbf{1}_{B(x,4s)} H_s\right\|_2^2 s^{\beta-n}\,ds.$$

We finally turn to J_j, for $j \geq 1$. For fixed $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{B(x,t)}(y) \left| \int_{t/2}^t \frac{\psi^{-,\ell}_{DB_0}(t,s)}{(s-t)} \left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y)\,ds \right|^2 t^{\beta-n}\,dy\,dt \\
&\leq \iint_{\mathbb{R}^{1+n}_+} \mathbf{1}_{B(x,t)}(y) \left(\int_{t/2}^t \left|\psi^{-,\ell}_{DB_0}(t,s)\left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y)\right| \frac{ds}{t-s}\right)^2 t^{\beta-n}\,dy\,dt \\
&\lesssim \int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{B(x,t)}(y) \int_{t/2}^t \left|\psi^{-,\ell}_{DB_0}(t,s)\left(\mathbf{1}_{C_j(x,4s)} H_s\right)(y)\right|^2 \frac{ds}{(s-t)^2} t^{\beta-n+1}\,dy\,dt \\
&\lesssim \int_0^\infty \int_{t/2}^t (s-t)^{-2}\left(1+\frac{2^j t}{t-s}\right)^{-2\ell} \left\|\mathbf{1}_{B(x,2^{j+2}s)} H_s\right\|_2^2 s^{\beta-n+1}\,ds\,dt \\
&\lesssim 2^{-2j\ell} \int_0^\infty \left(\int_s^{2s} s(s-t)^{-2}\left(1+\frac{2^j t}{t-s}\right)^{-2} dt\right) \left\|\mathbf{1}_{B(x,2^{j+2}s)} H_s\right\|_2^2 s^{\beta-n}\,ds \\
&\lesssim 2^{-2j\ell} \int_0^\infty \left\|\mathbf{1}_{B(x,2^{j+2}s)} H_s\right\|_2^2 s^{\beta-n}\,ds,
\end{aligned}$$

where we have used Cauchy-Schwarz inequality in the second inequality, the off-diagonal estimates (and local coercivity inequality), and the fact that $s \leq t$ in the third, Fubini's theorem and the fact that $s \geq \frac{t}{2}$ in the fourth, and the change of variable $\sigma = \frac{t}{t-s}$ in the last. An application of the change of angles in tent spaces, then gives

$$J_j \lesssim 2^{-j\ell} 2^{j\frac{n}{\tau}} \|H\|_{T^p_\beta} = 2^{-j\left(\ell - \frac{n}{\tau}\right)} \|H\|_{T^p_\beta},$$

and the proof is concluded by summing the estimates.

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